In this course, we'll study Heegaard Floer homology, a variant of Lagrangian Floer homology defined by Ozsváth and Szabó in 2001.

The main input data (for the 3-mfld invariants) is a Heegaard diagram for $Y^3$ (closed, oriented, connected), a set of data describing a Heegaard splitting.

The rough definition:
Given a Heegaard diagram $(\Sigma, \alpha, \beta)$ for $Y^3$ (i.e. $Y = \sum \alpha_1 D^2 u \alpha_2 D^2 v \ldots \alpha_g D^2 u \beta_1 D^2 v \ldots \beta_g D^2 v$), we define the $g$-fold symmetric product $\text{Sym}^g(\Sigma) := \frac{\Sigma \times \ldots \times \Sigma}{S_g}$ and study tori $T_\alpha = \alpha_1 \times \ldots \times \alpha_g \subset \text{Sym}^g(\Sigma)$ and $T_\beta = \beta_1 \times \ldots \times \beta_g$.

Then $\text{CF}(T_\alpha, T_\beta) := \mathbb{Z} \langle T_\alpha \cap T_\beta \rangle$, $\mathcal{D}$ counts "pseudoholomorphic curves" and $H_*^\mathfrak{F}(T_\alpha, T_\beta) := H_*^\mathfrak{F}(\text{CF}(T_\alpha, T_\beta), \mathcal{D})$. 

\text{1}
Heegaard Splittings: Let \( Y^3 \) be a closed, oriented, connected 3-mfd.

Def: A Heegaard splitting for \( Y \) is a decomposition
\[
Y^3 = U_0 \cup_f U_1,
\]
where \( U_i \) are handlebodies of genus \( g \) and \( f: \partial U_0 \to \partial U_1 \) is a diffeomorphism.

Theorem: Every such \( Y^3 \) admits a Heegaard splitting.

Pf: We use that every \( Y^3 \) can be triangulated (theorem of Moise).

Let \( \Gamma_i \) = 1-skeleton of triangulation
\( \Gamma'_2 \) = dual graph (vertex for each face, edge for each face identification)

Then \( Y = U_1 \cup U_2 \), where \( U_i = N(\Gamma_i) = \text{nbhd of } \Gamma_i \)

Claim: \( U_i \) are handlebodies

Let \( T_i \subset \Gamma_i \) be a maximal tree. Then \( N(T_i) \approx \mathbb{B}^3 \)
and adding edges to \( T_i \) corresponds to attaching handler to \( N(T_i) \):

\[\text{Diagram of handlebodies}\]
Now we can describe a handlebody $U_\alpha$ by $\alpha_1, \ldots, \alpha_g$, a tuple of s.c.e. disjointly embedded curves in $\partial U_\alpha$ bounding disks in $U_\alpha$ with $[\alpha_1], \ldots, [\alpha_g]$ L.I. in $H_1(\partial U_\alpha) \cong \mathbb{Z}^{2g}$, i.e., $U_\alpha = \Sigma \cup D \cup \ldots \cup D$

So, a Heegaard splitting $Y^3 = U_\alpha \sqcup U_\beta$

can be described by a Heegaard diagram $(\Sigma, \alpha, \beta)$, where

- $\alpha = (\alpha_1, \ldots, \alpha_g)$, $\alpha_i$ s.c.e. embedded disjointly in $\Sigma$, L.I. in $H_1(\Sigma)$
- $\beta = (\beta_1, \ldots, \beta_g)$, $\beta_i$ """"""""
- $\alpha_i \cap \beta_j$
- Each $\alpha_i$ bounds a disk in $U_\alpha$
- Each $\beta_i$ """""""" $U_\beta$

But a particular Heegaard splitting can be represented by many diagrams; consider the inclusion $i: \Sigma \to U_\alpha$

Then there's a diagram for each choice of basis for $\ker(i_*)$,

$$i_*: H_1(\Sigma) \to H_1(U_\alpha)$$

$$\mathbb{Z}^{2g} \to \mathbb{Z}^g$$
Different tuples $\alpha$ are related ( upto isotopy) by handleslides:

$$[\alpha_i] \rightarrow [\alpha_i] \pm [\alpha_j] \quad (j \neq i)$$

Geometrically:

$\alpha_i \quad \alpha_j$ $\rightarrow$ 

$\alpha_i \quad \alpha_j$

Move concretely:

$\alpha_1 \quad \alpha_2$ $\rightarrow$ $\alpha_1 \quad \alpha_2$

So, we can view

$$\{ \text{Heegaard splittings} \} = \{ \text{Heegaard diagrams} \} / \text{isotopy, } h\text{-slide.}$$

One can in fact compute $H_*(Y)$ using He diagrams:

$$H_0(Y) \cong H_3(Y) \cong \mathbb{Z}$$

$$H_2(Y) \cong H_1(Y) = \text{Hom} (H_1(Y), \mathbb{Z}) \oplus \text{Tor} (H_0(Y), \mathbb{Z})$$

$$= \mathbb{Z}^{b_1} \quad (b_1 = \dim_{\mathbb{Q}} (H_1(Y; \mathbb{Q})))$$

So, only need to compute $H_1(Y)$:

Exercise: $H_1(Y) \cong H_1(\Sigma) / \langle [\alpha], [\alpha_9], [\beta], \ldots, [\beta_g] \rangle$

Hint: Mayer-Vietoris.
Remark: We’ll often draw Heegaard diagrams in the plane $-\Sigma_g$ is constructed from $S^2 = \mathbb{C} \cup \mathbb{C} \cup \mathbb{C}$ by attaching $g$ 1-handles, where we draw attaching spheres as pairs of matching disks, i.e.

$$\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array} = \begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array} = \begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array} = \begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array} \cup \begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array} \cup \begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array}$$

$\{\alpha\}$

E.g. ($g=0$) $\Sigma=\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array}$, $\alpha=\beta=\emptyset \rightarrow B^3 \cup B^3 = S^3$

$\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array}$

Notice: $H_1(S^3) = \langle \alpha, \beta \rangle / \langle \alpha, \beta \rangle = 0$

$\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array}$

Indeed, $H_1(Y) = \langle \alpha, \beta \rangle / \langle \alpha \rangle \cong \mathbb{Z}$

(c) More generally, we can classify genus-1 examples. Upto diffeomorphism of $\Sigma$, we can choose $\alpha$ to be as in

$$\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\hline
\alpha & \beta & \gamma & \delta \\hline
\end{array}$$

$\beta$ can be anything of the form $\beta = pm + q\alpha$, $\gcd(p, q) = 1$. 
Assume $\alpha \cdot m = +1$, $m \cdot a = -1$, $m \cdot m = \alpha \cdot a = 0$

Then $\beta \cdot a = -p$, $\beta \cdot m = q$ (where $p, q$ determine $\beta$) \(\text{upto isotopy}\)

e.g. $p = 5$, $q = -2$

The associated 3-mfld is the lens space $L(p,q)$

Alternatively,

\[
L(p,q) = S^3 / Z_p = \left\{ (z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \right\} / Z_p
\]

where the action of $u \in Z_p$ is $(z,w) \mapsto (uz, u^q w)$

$S^3$ is a universal cover for $L(p,q)$, so

\[
\pi_1(L(p,q)) \cong H_1(L(p,q)) \cong \mathbb{Z}_p
\]

(Heegaard diagram: $H_1(Y) = \langle \alpha, m \rangle / \langle \alpha, pm + q\alpha \rangle \cong \langle m \rangle / \langle pm \rangle \cong \mathbb{Z}_p$)
We can also adjoin Heegaard diagrams:

\[ (\Sigma, \alpha, \beta), (\Sigma', \alpha', \beta') \mapsto (\Sigma \# \Sigma', \alpha \cup \alpha', \beta \cup \beta') \]

\[ Y \mapsto Y' \]

In particular, \((\Sigma, \alpha, \beta) \# (\Sigma')\) represents \(Y \# S^3 = Y\).

This operation is called **stabilization**.

**Thm (Reidemeister/Singer):** Any two Heegaard splittings for the same 3-mfd have a common stabilization (may need to stabilize each many times)

So,

\[
\{ \text{H. splittings}^3 \}/\text{stabilization} = \{ \text{3-mfds}^3 \}
\]

and

\[
\{ \text{3-mfds}^3 \} = \{ \text{H. diagrams}^3 \}/\text{isotopy, h-slide, stabilization}
\]

The plan: Do Floer homology using \((\Sigma, \alpha, \beta)\) for \(Y\), but verify that \(HF(\Sigma, \alpha, \beta)\) only depends on \(Y\) \(\Rightarrow 3\)-mfd invariant \(HF(Y)\).
Preliminaries on Symplectic Topology:

Def: Let $M^{2n}$ be a smooth mfd and let $\omega \in \Omega^2(M, \mathbb{R})$ be such that

- $d\omega = 0$ (\(\omega\) closed) and
- $\omega \wedge \ldots \wedge \omega \in \Omega^{2n}(M, \mathbb{R})$ is nowhere-vanishing (\(\omega\) non-degenerate)

Then $(M, \omega)$ is a symplectic manifold.

e.g. 1) $(\Sigma, dA)$ is symplectic

(\(\Sigma\) any Riemann surface

2) $(\mathbb{R}^{2n}, \omega_{\text{can}})$, where \(\mathbb{R}^{2n}\) has coordinates $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$

\[
\omega_{\text{can}} = \sum_{i=1}^{n} dx_i \wedge dy_i
\]

Def: A submanifold $L^n \subset (M^{2n}, \omega)$ is called Lagrangian if $\omega|_L = 0$.

e.g. 1) Any curve in $\mathbb{R}^2$

2) Any curve on $\Sigma$

3) $\mathbb{R}^n \subset \mathbb{R}^{2n}$

4) (non-example): $V^n \subset (\mathbb{R}^{2n}, \omega_{\text{can}})$

\[
V^n = \left\{ (x_1, y_1, \ldots, x_n, y_n) \mid x_1 = y_2, \quad y_3 = y_4 = \ldots = y_n = y_1 = 0 \right\}
\]

Then $dx_2 \wedge dy_2 \big|_V = 0$. 

8
Def: An \textit{almost-complex} (a.c.) structure $J$ on $M$ is a bundle map $J: TM \to TM$ covering the identity on $M$ such that $J^2 = -I$

schematically: \[ \begin{array}{c}
J \\
\downarrow \\
M \\
\end{array} \quad T_x M \quad ("J = i") \]

Def: $J$ is \underline{compatible} with $\omega$ if
- $\omega(v, w) = \omega(Jv, Jw), \forall v, w$
- $\omega(v, Jv) > 0, \forall v \neq 0$

Rmk: Given $J$ compatible with $\omega$, $g_f(v, w) := \omega(v, Jw)$ is a Riemannian metric on $M$.

We denote by $\mathcal{J}(M, \omega)$ the space of $J$ on $M$ compatible with $\omega$.

Proposition: $\mathcal{J}(M, \omega)$ is nonempty and contractible.

Sketch of $\mathcal{J}(M, \omega)$: Will first define a continuous map $r: \text{Met}(M) \to \mathcal{J}(M, \omega)$ (Riemannian metrics).

Given $g$, since $g, \omega$ non-degenerate, $\exists ! A \in \text{End}(TM)$ s.t. $\omega(v, Aw) = g(v, w)$. One can verify that $A^* = -A$, and that $P := AA^*$ is self-adjoint and pos. def. $\Rightarrow \exists !$ self-adjoint $Q$ s.t. $Q^2 = P$

Then one can verify that $J_g := Q^{-1}A \in \mathcal{J}(M, \omega)$, we let $r(g) := J_g$. \( \Box \)
One can verify that $r$ is continuous, and in fact that $r(g_J) = J \ (\forall J \in \mathcal{J}(M,\omega))$

Then define a retract to some fixed $J_0$ via

$$f_t(J) = r((1-t)g_J + tgt_{J_0}), \ t \in [0,1]$$

We'll study certain types of maps into $M$:

**Def:** Let $\Sigma$ be a Riemann surface equipped with a complex structure $J$; and let $(M,\omega)$ have $J \in \mathcal{J}(M,\omega)$. Then $u: \Sigma \to M$ is a $J$-holomorphic curve if $du \circ J = J_0 du$

(equivalently, $\bar{\partial}_J u = 0$, where

$$\bar{\partial}_J u := \frac{i}{2} (du + J_0 du \circ J)$$

We'll be interested in three types of $J$-holomorphic curves:

1. **Spheres:** $u: S^2 \to M$
2. **Disks:** $u: (D^2, \partial D) \to (M,\omega)$ (L Lagrangian)
3. **Strips:** $L_0, L_1$ Lagrangian, $x, y \in L_0 \cap \overline{L_1}$

$u: \mathbb{R} \times [0,1] \to M$ such that

- $u(s, 0) \in L_0$, $u(s, 1) \in L_1$
- $\lim_{s \to -\infty} (u(s,t)) = x$, $\lim_{s \to \infty} (u(s,t)) = y$. 

(10)
Def: If $u: \Sigma \to M$ is some smooth map, the energy of $u$ is

$$E(u) = \int_{\Sigma} |du|^2 \text{dvol}_\Sigma$$

(if $(e_1, e_2)$ is an orthonormal frame for $g_J$, then)

$$|du|^2 = |du(e_1)|^2_{g_J} + |du(e_2)|^2_{g_J}$$

Proposition: $E(u) = \int_{\Sigma} u^* \omega + \int_{\Sigma} |\overline{\partial}u|^2 \text{dvol}_\Sigma$

Pf: See exercises.

* In particular, if $u$ is $J$-holomorphic, $E(u) = \int_{\Sigma} u^* \omega$.

Question: Do sequences of holomorphic maps have limits?  

Theorem: Let $J \in J(M, \omega)$, and let $\{u_n\}$ be a sequence of $J$-holomorphic objects of a particular type (spheres, disks, or strips). Assume that $\exists M$ with $E(u_n) < M$, $\forall n$. Then:

(a) $u_n: S^2 \to M \Rightarrow \exists$ Subsequence $\mathcal{I} \subseteq \mathcal{N}$ s.t.

$$u_{\mathcal{I}_n} \to u, \text{ a finite bubble free of spheres: } \bigcirc, \bigcirc, \bigcirc, \bigcirc \rightarrow \bigcirc$$
(b) \( u_n : (D^3, \partial D) \to (M, L) \Rightarrow \exists\) subsequence
\[ \{u_{n_k}\}, \quad u_{n_k} \to u, \text{ a tree of disks and spheres:} \]

(C) \( u_n : \mathbb{R} \times [0, 1] \to M \) (with strip boundary conditions)
\[ \Rightarrow \exists\) subsequence \( \{u_{n_k}\}, \quad u_{n_k} \to u, \text{ a broken strip with disk and sphere bubbles:} \]

By Riemann mapping theorem, there's a holomorphic strip \( u_r : \mathbb{R} \times [0, 1] \to \text{ } \) for each \( r \in [0, R) \)
As \( r \to R, \quad u_r \to u, \text{ the broken strip } (\text{ broken strip diagram }) \)
Remark:
From here on out, we'll precompose with the usual conformal map $D^2 \rightarrow \mathbb{R} \times [0,1]$ s.t.
- $S^1 \cap \{ \text{Re}(z) \geq 0 \} \rightarrow \mathbb{R} \times \{0\}$
- $S^1 \cap \{ \text{Re}(z) \leq 0 \} \rightarrow \mathbb{R} \times \{1\}$
- $\lim_{s \rightarrow -\infty} f^{-1}(s,t) = -i$
- $\lim_{s \rightarrow \infty} f^{-1}(s,t) = i$

So, we'll instead study:

**Def:** A Whitney disk from $x \to y$ ($x,y \in L_0 \cap L_1$) is a smooth map $u: D^2 \rightarrow M$ such that
- $u(S^1 \cap \{ \text{Re}(z) \geq 0 \}) \subset L_0$
- $u(S^1 \cap \{ \text{Re}(z) \leq 0 \}) \subset L_1$
- $u(-i) = x$
- $u(i) = y$

The set of homotopy classes of Whitney disks from $x \to y$ will be denoted by $\pi_2(x,y)$.

**Def:** Let $\{ J_t \}$ be a continuous path in $\mathcal{J}(M,\omega)$.
$u: \Sigma \rightarrow M$ is $J_t$-holomorphic if $\frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} = 0$

(let now $\Sigma$ varies with $t$)

Let $M_{J_t}(\phi)$ denote the moduli space of $J_t$-hol. Whitney disks $u$ with $[u] = \phi \in \pi_2(x,y)$. 

13
Theorem: For generic \( \{ J^+_t \} \), \( M^+_t(\phi) \) is a smooth manifold.

("generic" means the good \( \{ J^+_t \} \) constitute a second category Baire set in the space of all families)

Def: For \( L_0, L_1 \in C(M,\omega) \) Lagrangians, \( \{ J^+_t \} \subset J(M,\omega) \), \( L_0 \neq L_1 \), \( x, y \in L_0 \cap L_1 \), and \( \phi \in \pi_2(x, y) \), the Maslov index of \( \phi \), \( m(\phi) \in \mathbb{Z} \), is the index of \( \delta^+_t \) near \( u \) with \( [u] = \phi \).

Fact: 1) When \( M^+_t(\phi) \neq \emptyset \), \( \dim(M^+_t(\phi)) = m(\phi) \).

(for generic \( \{ J^+_t \} \))

2) When \( M^+_t(\phi) \neq \emptyset \), it carries an \( IR \)-action (coming from precomposing with a shift on \( \equiv \)). This is free and transitive as long as \( \phi \) is not the class of the constant map. So,

- If \( \phi \) is not constant and \( M^+_t(\phi) \neq \emptyset \), then \( M^+_t(\phi) / IR \) is a smooth mfd. of dimension \( m(\phi) - 1 \).

- In particular, \( m(\phi) \geq 1 \) in this case.
There are two "cuts" we can make (but can't make both and still obey boundary conditions):

\[ M_\phi (\phi) \cong (\mathbb{R} \times (-R_1, 0]) \cup (\mathbb{R} \times [0, R_2]) \]

\[ M_\phi (\phi) \cong \mathbb{R} \times (-R_1, R_2) \]

So, \( \mu(\phi) = 2 \)

Recall: \( \pi_2(x, y) = \left\{ \text{homotopy classes of Whitney disks from } x \text{ to } y \right\} \)

There are two operations:

1. Induces concatenation operation \( \ast : \pi_2(x, y) \times \pi_2(y, z) \to \pi_2(x, z) \)

2. \( f : D^2 \to D^2, \ f(z) = \overline{z} \) induces "\(-\)" \( \pi_2(x, y) \to \pi_2(y, x) \)

(both arise by precomposing maps with the indicated functions)
Facts:
1. \( \mu(\phi_1 \times \phi_2) = \mu(\phi_1) \cdot \mu(\phi_2) \)
2. \( \mu(\phi) = -\mu(\phi) \) (in particular, if \( \mu(\phi) \geq 1 \), then \( M_{\mathcal{F}}(\phi) = \emptyset \))
3. \( \sum \omega^*(u_1 \times u_2) = \sum \omega^* u_1 + \sum \omega^* u_2 \)
   (in particular, if \( u_1, u_2 \) are \( \mathcal{F} \)-holomorphic, \( E(u_1 \times u_2) = E(u_1) + E(u_2) \))

\[ \begin{array}{c}
\text{e.g.:} \\
\begin{array}{c}
\text{\includegraphics[width=10cm]{example.png}} \\
( \omega^* (u_1 \times u_2) = \omega^* u_1 \times \omega^* u_2 = \{ (u_1, u_2) \} \text{ and } \omega^* (u_1 \times u_2) = \omega^* u_1 \times \omega^* u_2 = \{ (u_1, u_2) \})
\end{array}
\end{array} \]

Notice that \( \pi_2(x, x) \) acts freely and transitively on \( \pi_2(x, y) \) and \( \pi_2(y, x) \). Further, \( \pi_2(x, x) \) is a group with respect to "\(*\)"

\( u \times (-u) \) is homotopic to the constant map)

There is a homomorphism \( M: \pi_2(x, x) \rightarrow \mathbb{Z} \),
and \( \text{im}(M) = N_x \mathbb{Z} \).

Exercise: If \( \pi_2(x, y) \neq \emptyset \), \( N_x = N_y = N \).
Let $L_0, L_1 \subset (M, \omega)$ be Lagrangians, and assume:

1. $L_0, L_1$ cpt.
2. $L_0 \cap L_1$
3. $M$ is cpt. or "convex at infinity", so no hol. curves go off to $\infty$ w/ boundary on a cpt. set.
4. $[w](A) = 0$ for all $A \in \Pi_2(M)$ and all $A \in \Pi_2(M, L_i)$
   ($\Rightarrow$ No $J_{\epsilon}$-hd. spheres or disk, only strips)
5. $E(\ker (u_x : \pi_2(x, x) \to \mathbb{Z}))$ is bounded above

($\Rightarrow$ For each $x, y \in L_0 \cap L_1$ with $\pi_2(x, y) \neq \emptyset$, there is a universal bound on holomorphic strips $\mu$ with $[\mu] \in \pi_2(x, y)$)

---

1. Floer chain complex: $CF_*(L_0, L_1) := \mathbb{Z} \langle L_0 \cap L_1 \rangle$
   * $L_0 \cap L_1$ finite, since $\dim L_0 + \dim L_1 = \dim M$ and $L_0 \cap L_1$.

2. Gradings:
   a. $\mu_x : \pi_2(x, x) \to \mathbb{Z}$ is a group homomorphism; $\text{im \mu}_x = N_x \mathbb{Z}$
   \[ (\pi_2(x, y) \neq \emptyset \Rightarrow N_x = N_y = N) \]
   \[ \Rightarrow \text{For } \Psi \in \pi_2(x, y), \text{ relative grading } g(\Psi) = \mu(\Phi) \pmod{N} \]
   b. If $L_0, L_1$ oriented, can define $\sigma : L_0 \cap L_1 \to \mathbb{Z}/2\mathbb{Z}$ (sign)
   Then if $\Psi \in \pi_2(x, y)$, $\mu(\Psi) \equiv \sigma(x) - \sigma(y) \pmod{2}$
   \[ \Rightarrow N \text{ is even} \]
3. Equivalence classes:

Define an equivalence relation on \( L_0 \cup L_1 \):

\[ (x \sim y) \iff \pi_2(x,y) \neq \emptyset \]

Denote the set of equivalence classes by \( \mathcal{E}(L_0, L_1) \).

4. Differential:

\[
\mathcal{D} : C^L(L_0, L_1) \to \mathbb{R}, \quad \mathcal{D} x = \sum_y \sum_{\phi \in \pi_2(x, y)} \#(M_{\mathcal{D}}(\phi)/\mathbb{R}) y
\]

**Theorem (Floer):** Under our assumptions on \((M, \omega), L_0, L_1\),

\[ M(xy) := \bigsqcup_{\phi \in \pi_2(xy)} M(\phi)/\mathbb{R} \]

is a smooth mfd of dimension zero.

(when its non-empty)

**Proposition:** When \( M(xy) \neq \emptyset \), it's already cpt.

**Proof:** Let \( \{u_n\}_n \) be a sequence in \( M(xy) \). There's a uniform bound on \( E(u_n) \), so by Gromov compactness theorem, \( \exists \) subsequence \( \{u_{n_i}\}_i \)

with \( u_{n_i} \to u \), a broken strip \( u = v \times w (w : z \to y) \)

Now \( \mu([v]) + \mu([w]) = 1 \) (since \( \mu([u_n]) = 1 \)), so one of \([v]\) or \([w]\) has \( \mu \leq 0 \). But then either \( x \equiv z \) and \( v \) = constant map or \( z = y \) and \( w \) = constant map.

So, \( u \) is an honest (unbroken) strip and \( u \in M(xy) \).
So, when $M(x,y) \neq \emptyset$, it's a finite number of points. ($\Rightarrow$ Sums for $\Delta x$ are finite)

Rmk: We've suppressed orientation, but there's a way to orient $M(y)$ so the points in $M(x,y)$ are signed.

**Proposition:** $\partial^2 = 0$

**Proof:** $\partial^2 = \partial \left( \sum \sum \# \left( M(\phi) \cap \mathbb{R} \right) \right) = \sum \sum \sum \sum \left[ \# \left( M(\phi) \cap \mathbb{R} \right) \cdot \# \left( M(\psi) \cap \mathbb{R} \right) \right] \neq 0$

$= \sum \sum \sum \left[ \# \left( M(\phi) \cap \mathbb{R} \right) \cdot \# \left( M(\psi) \cap \mathbb{R} \right) \right]$

Well, $(\ast) = \# \left( \bigcap \left( M(\phi) \cap \mathbb{R} \right) \times \left( M(\psi) \cap \mathbb{R} \right) \right)$

and $\ast$ is the boundary of the cpt $1$-dim oriented smooth $mfcl \overline{M(\phi)/\mathbb{R}}$. So, $(\ast) = 0$. (e.g. $\# \left( 00 + 00 - 00 + \right) = 0$)
We can then define Lagrangian Floer homology as \( HF_*(L_0, L_1) := H_*(CF(L_0, L_1), \partial) \).

**Proposition:** \( HF_*(L_0, L_1) \) is independent of the choice of generic \( \xi \in \mathcal{J}_t \) \( \mod 2 \).

**Pf:** Let \( J_t^0, J_t^{+1} \) be two generic families.

Interpolate between them via \( J_t^s \) \((s \in (-\infty, \infty))\), where \( J_t^s = J_t^0 \) for \( s < 0 \), \( J_t^s = J_t^{+1} \) for \( s \geq 1 \).

Then define \( M_{J_t^s}(\phi) := \text{moduli space of } J_t^s \text{-hol. strips in the class } \phi \) \( (J_t^s \text{-hol. means } \frac{\partial u}{\partial s} + J_t^s \frac{\partial u}{\partial t} = 0 \Rightarrow \text{no a priori } IR \text{-action!}) \).

Will define chain maps \( F, G \) and homotopies \( H, M \):

\[
\begin{align*}
H : (CF, \partial_0) &\xrightarrow{F} (CF, \partial_1) \xleftarrow{G} M
\end{align*}
\]

\( \partial_0 H + H \partial_0 = G \circ F + \text{id}_0 \), \( \partial_1 M + M \partial_1 = F \circ G + \text{id}_1 \)

Let \( F(x) = \sum_x \sum_{\xi} \#(M_{J_t}^s(\phi)) y \left( \begin{array}{c} G \text{ counts } M_{J_t}^{s-1} \text{ instead} \end{array} \right) \)

\( M_{J_t}^s(\phi) = 0 \)

Claim: \( F, G \) are chain maps (will show for \( F \) only)

The coeff. of \( z \) in \( \partial F(x) \) counts \( x \xrightarrow{J_t^s} y \xrightarrow{J_t} z \), and

\( \partial F(x) = x \xrightarrow{J_t^0} y \xrightarrow{J_t^s} z \), and

\( \partial F(x) = x \xrightarrow{J_t^0} y \xrightarrow{J_t^s} z \)
But these (collectively) make up the boundary of
\[ \bigcup_{\phi \in \Pi_t(x,y), m(\phi) = 1} M_{J^s(\phi)}(\phi) \] a smooth cpt mfd dimension 1
So, \( \partial F + F \partial = 0. \)

Now let \( J^s_t \quad (t \in [0,1]) \) be a homotopy connecting \( J^s_0 = J^0_t \) (independent of \( s \)) to \( J^s_1 = J^s_t * J^{-s}_t \) (juxtaposition of \( J^s_t \) and \( J^{-s}_t \))

\[ \bigcup_{t \in [0,1]} \bigcup_{\phi \in \Pi_t(x,y), m(\phi) = k} M_{J^s_t}(\phi) \] is a \((k+1)\)-dimensional smooth manifold, and we let

\[ H(x) = \sum_{t \in [0,1]} \sum_{\phi \in \Pi_t(x,y), m(\phi) = -1} \# \left( \bigcup_{t \in [0,1]} M_{J^s_t}(\phi) \right) \left( \begin{array}{c} M \text{ counts} \\ M_{J^s_t, 1-t} \end{array} \right) \]

\((\ast)\)

(Now \((\ast)\) is generically empty \((m(\phi) = -1), \) but one can arrange that for a finite \# of values of \( t \), it becomes a cpt \( 0 \)-mfd.

Notice: The coeff of \( z \) in \((G \circ F)(x)\) counts:
\[ \begin{array}{l}
  \mu = 0 \\
  \mu = 0
\end{array} \]

\[ \begin{array}{l}
  \mu = 1 \\
  \mu = -1
\end{array} \]

\[ \begin{array}{l}
  \mu = -1 \\
  \mu = 1
\end{array} \]

\[ \begin{array}{l}
  \mu = 0 \\
  \mu = 0
\end{array} \]

Notice: The coeff of \( z \) in \((G \circ F)(x)\) counts:
\[ \begin{array}{l}
  X \\
  X, Y
\end{array} \]

\[ \begin{array}{l}
  X \\
  X, Y
\end{array} \]

\[ \begin{array}{l}
  X \\
  X, Y
\end{array} \]

\[ \begin{array}{l}
  \emptyset \\
  \emptyset
\end{array} \]

21
Collectively, these constitute the boundary of \[ \bigcup_{t \in [a, b]} \mathcal{M}_{s,t}(\phi), \]
where \( \mathcal{M}(\phi) = 0 \) and \( \phi \in \mathcal{T}(x,v) \),
resulting in a compact 1-dimensional smooth manifold.

Now our goal is to do Lagrangian Floer homology with \( M = \text{Sym}^g(\Sigma) \) (\( \text{Sym}^g(\Sigma ; z) \) for \( H\tilde{F} \))
and \( L_0 = \Pi_\alpha = \alpha_1 \times \ldots \times \alpha_g \), \( L_1 = \Pi_\beta = \beta_1 \times \ldots \times \beta_g \)

[here \( (\Sigma, \alpha, \beta, z) \) is a pointed Heegaard diagram
for some closed, oriented, connected 3-manifold \( Y \) –
\( (\Sigma, \alpha, \beta) \) is a Heegaard diagram and \( z \in \Sigma \setminus (\alpha \cup \beta) \)
is a distinguished basepoint.]

Recall: \( \text{Sym}^g(\Sigma) = \frac{\Sigma \times \ldots \times \Sigma}{\mathbb{S}_g} \) (permutes the factors)

We'll now take a short detour to study symmetric products
in particular, before we claim \( \text{Sym}^g(\Sigma) \)
is a symplectic manifold, we'll need to show that it's a manifold at all! The
action of \( \mathbb{S}_g \) isn't free...
Def: Let $M$ be a smooth manifold. Then the $k$-fold symmetric product of $M$ is

$$\text{Sym}^k(M) := \frac{M \times \ldots \times M}{S_k}$$

where the $S_k$-action permutes the factors of $M$.

$\text{Sym}^k(M)$ is not a priori a manifold, since the $S_k$-action is not free.

Example: $\text{Sym}^0(M) = \{\text{pt}\}$
$\text{Sym}^1(M) = M$

How about $\text{Sym}^2(M)$? Let's look locally.

If $m = (m_1, m_2) \in \text{Sym}^2(M)$, then local model is $\mathbb{R}^d \times \mathbb{R}^d$

If $m_1 = m_2$, then local model is $\text{Sym}^2(\mathbb{R}^d)$

Let's study $\text{Sym}^2(\mathbb{R}^d)$:

$$\text{Sym}^2(\mathbb{R}^d) = \frac{\mathbb{R}^d \times \mathbb{R}^d}{(x, y) \sim (y, x) \sim (x+y, v)}$$

$$= \mathbb{R}^d \times \left( \frac{\mathbb{R}^d}{v \sim (-v)} \right)$$

Now recall that $\text{Cone}(X) := X \times [0, \infty) / X \times \{0\}$
(for $X$ a top. space)

In particular, $\mathbb{R}^d \cong \text{Cone}(S^{d-1})$

So, $\mathbb{R}^d / v \sim (-v) \cong \text{Cone}(S^{d-1}) / v \sim (-v) = \text{Cone}(S^{d-1} / v \sim (-v)) = \text{Cone}(\mathbb{R}^{d-1})$

* When $d = 2$, $\text{RIP}^{d-2} \cong \text{RIP} \cong S^1$

and so $\text{Cone}(\text{RIP}) = \mathbb{R}^2 = \mathbb{C}$
So, \( \text{Sym}^2(\mathbb{R}^2) = \text{Sym}^2(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C}^{\text{coordinate}} \),

so, \( \text{Sym}^2(M) \) is a manifold when \( \dim M = 2 \).

**Proposition:** When \( \Sigma \) is a Riemann surface, 
\( \text{Sym}^k(\Sigma) \) is a smooth mfd of dimension \( 2k \).

**Pf:** Let \( x \in \text{Sym}^k(\Sigma) \), and assume 

\[
X = \{ x_1, \ldots, x_k, x_2, \ldots, x_k, \ldots, x_n, \ldots, x_n \} \quad \text{with} \quad x_i + x_j \text{ when } i \neq j.
\]

Then the local model near \( x \) is 
\( \text{Sym}^k(\mathbb{C}) \times \ldots \times \text{Sym}^n(\mathbb{C}) \) (where \( x \rightarrow (x_0, \ldots, x_n, \ldots) \)).

It suffices to identify \( \text{Sym}^k(\mathbb{C}) \) with a known smooth manifold — this manifold will be \( \mathbb{C}^2 \).

Let \( \{ z_1, z_2, \ldots, z_k \} \in \text{Sym}^k(\mathbb{C}) \).

Define 
\[
S_1 = z_1 + \ldots + z_k
\]
\[
S_2 = \sum_{i \neq j} z_i z_j
\]
\[
\vdots
\]
\[
S_m = \sum_{i_1, \ldots, i_m \text{ distinct}} z_{i_1} \ldots z_{i_m}
\]
\[
S_k = z_1 z_2 \ldots z_k
\]

Then the \( z_i \) are exactly the roots of 
\[
\prod_{i=1}^{k} (t - z_i) = t^k - s_1 t^{k-1} + s_2 t^{k-2} - \ldots + (-1)^k s_k
\]

\[ \therefore \]
Now we return to studying $\text{Sym}^g(\Sigma)$, where $g = \text{genus}(\Sigma)$.
($\Sigma$ arises in a Heegaard diagram, and is closed)

Let $\Delta \subset \Sigma^g$ be $\Delta := \{(x_1, \ldots, x_k) | x_i = x_j \text{ for some } i \neq j\}$
($\Delta$ is the "fat diagonal")

Then let $\Delta := \Delta / S_g$; the action of $S_g$ is free on $\Sigma^g \setminus \Delta$.

Note: $T_a, T_\beta \subset \text{Sym}^g(\Sigma) \setminus \Delta \Rightarrow$ they're actually tori in $\text{Sym}^g(\Sigma)$.

Given some a.c. structure $j$ on $\Sigma$, can induce a.c. structure $\text{Sym}^g(j)$ on $\text{Sym}^g(\Sigma)$

in the obvious way

(will need to use identification $\text{Sym}^g(C) \leftrightarrow C^g$)

What we'd hope: Given the area form $\omega := dA$

on $\Sigma$, that $\text{Sym}^g(\omega)$ is symplectic.

Let $p_i : \Sigma^g \rightarrow \Sigma$ be the projection map. Then consider the symplectic form $\tilde{\omega} := p_1^* \omega + p_2^* \omega + \ldots + p_g^* \omega$ on $\Sigma^g$.

Notice $\tilde{\omega}$ is $S_g$-invariant; indeed we can push forward $\tilde{\omega}$ to get symplectic form on $\text{Sym}^g(\Sigma) \setminus \Delta$.

What happens near $\Delta$?
e.g. \( g = 2 \)

Recall: \( \text{Sym}^2(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C} / (z \sim (-z)) \).

\( \mathbb{C} / (z \sim (-z)) \cong \mathbb{C} \) (with coordinate \( z^2 \))

The area form on \( (\mathbb{C}, z) \) is \( dA = dx \wedge dy = \rho \, dp \wedge d\theta \)

\( z = x + iy = \rho e^{i\theta} \)

The area form on \( (\mathbb{C}, z^2) \) is \( dA' = \rho' \, dp' \wedge d\theta' \)

but \( z^2 = \rho^2 e^{2i\theta} \Rightarrow \rho' = \rho^2, \theta' = 2\theta \)

so, \( dA = \rho \, dp \wedge d\theta = \rho \left( \frac{\rho^2}{2} \right) \wedge d\theta = \frac{d\rho \wedge d\theta'}{4} = \frac{dA'}{4} \rho \)

So, \( dA \) blows up near \( z = 0 \) (i.e. \( z = (-z) \))

So, the push-forward of \( \tilde{\omega} \), \( \pi_x^* \tilde{\omega} \), blows up near \( \Delta \).

---

However, this can be remedied:

**Theorem (Perutz):**

\( \exists \) a symplectic form on \( \text{Sym}^3(\Sigma) \) which differs from \( \pi_x^* \tilde{\omega} \) only in a neighborhood of \( \Delta \).

Moreover, \( \text{Sym}^3(j) \) can be slightly perturbed to give \( J \in \mathcal{J}(\text{Sym}^3(\Sigma), \tilde{\omega}') \).

( won't prove this)
Now for $\hat{HF}$ (the simplest flavor of Heegaard Floer homology), we'll have $M = \text{Sym}^3(\Sigma \setminus z) = \text{Sym}^9(\Sigma) \setminus R_z$, where $R_z := \{z\} \times \text{Sym}^{g-1}(\Sigma) \subset \text{Sym}^g(\Sigma)$, a $(2g-2)$-dimensional submanifold.

Note: $\bar{\Pi}_\alpha, \bar{\Pi}_\beta$ avoid $R_z$.

**Def:** Given a Whitney disk $u : D^2 \to \text{Sym}^g(\Sigma)$, let $n_z(u) := \#(u^{-1}(R_z)) \sim \text{(counted with sign)}$ (generically, $\text{im}(u) \cap R_z$ is a finite # of points).

**Remarks:**

1. Descends to a map $n_z : \bar{\Pi}_z(x,y) \to \mathbb{Z}$
2. $n_z(\phi \times \psi) = n_z(\phi) + n_z(\psi)$
3. $n_z(-\phi) = -n_z(\phi)$
4. If $\phi$ has a holomorphic representative, then $n_z(\phi) \geq 0$ (holomorphic surfaces intersect non-negatively)
5. If $\mu(\phi) = 1$, $[u_h] = \phi$, $u_h \to u$, then we know $[u] = \phi \Rightarrow n_z(u) = n_z(u_h)$

So, sequences of holomorphic strips can't "go off to $\infty$" in $\text{Sym}^g(\Sigma) \setminus R_z$. 

\[ \text{(3)} \]
**Branched Covering Spaces**

**Def:** A branched covering space (with \( n \) sheets) is a map \( p : M^d \to N^d \), \( A, B \subset M \), \( A, B \subset N \) s.t. \( A, B \) are called the branch sets.

\[
\begin{align*}
(a) & \quad p(A) = B, \quad p(M \setminus A) = N \setminus B \\
(b) & \quad p|_{M \setminus A} \text{ is a } n \text{-fold covering space.}
\end{align*}
\]

**Note:** Typically it can be arranged that near \( x \in A \), \( p \) looks like \( \mathbb{R}^{d-2} \times \mathbb{C} \to \mathbb{R}^{d-2} \times \mathbb{C} \),

\[
(x_1, \ldots, x^{d-2}, z) \mapsto (x_1, \ldots, x^{d-2}, z^k)
\]

\( k \) is called the index of the branch point \( x \).

**Remark:** If \( y \in B \) and \( p^{-1}(y) = \{ \ldots \} \times \mathbb{S}_3 \),

then the indices of the \( x_i \) add up to \( n \).

(since each point \( y \in N \setminus B \) near \( y \) must have \( n \) distinct pre-images)

**e.g.**

- 2-fold branched covers of \( S^2 \) with \( k \) branch points of index 2 (assume orientable)

  Triangulate \( S^2 \) s.t. all branch points are vertices.

  Then \( 2 = \chi(S^2) = v - e + f \)
Now
\[ 2 - 2g = \chi(\Sigma) = v' - e' + f' \]
\[ v' = 2(v - k) + k \]
\[ e' = 2e \]
\[ f' = 2f \]
\[ \Rightarrow 2 - 2g = 2\chi(S^2) - k = 4 - k \Rightarrow \text{ } k \text{ is even and } \]
\[
\begin{align*}
g &= \frac{k - 2}{2} \\
g &\text{ explicitly: define an involution } \tau \text{ on } \Sigma_g : \\
\Sigma &:\begin{cases}
\tau \text{ rotates by } \pi \\
\end{cases}
\end{align*}
\]
\[ \frac{\Sigma}{\tau} = \begin{cases}
\text{glue} \\
\end{cases}
\]
\[ \text{i.e. } p: \Sigma \rightarrow \frac{\Sigma}{\tau} \text{ is the branched covering from before.} \\
( S^2 \text{ with } 2g + 2 \text{ branch points})
\]

2 Branched \( \Sigma \rightarrow D^2 \), the disk.
\( (n \text{ sheets, } k \text{ branch points in } D^2, \Sigma \text{ oriented}) \)
(a) Make these cuts in \( D^2 \): 
\[ \text{(let } x_1, \ldots, x_k \text{ be the branch points)} \]
\[ D^x := D \setminus \bigcup_{i=1}^{k} \{ x_i \}, \Sigma^x := \Sigma \setminus p^{-1}(\bigcup_{i=1}^{k} \{ x_i \}) \]
(b) The preimage of the punctured disk is a union of $n$ disks (glued in a certain way)

\[
\bigcup_{i=1}^{n} (a_i A_i b_i B_i c_i C_i) \rightarrow (\text{here } k=3)
\]

(c) downstairs we identify $a_i$ with $A_i$, etc. 
upstairs we must glue each $a_i$ to exactly one $A_j$; in other words, choose some $\sigma \in S_n$ and identify $a_i$ with $A_{\sigma(i)}$ for each $i$ (similarly for $b_i, c_i, \ldots$)

(d) Identify the surface $\Sigma$:

$\text{pl}_x: \Sigma^x \rightarrow \mathbb{D}^x$ is a covering map, so

$\chi(\Sigma^x) = n\chi(\mathbb{D}^x) = n(1-k)$

Now after performing gluings in (c), we can determine $M := |p^{-1}(x_1, \ldots, x_k)|$ ($m \geq k$)

Then $\chi(\Sigma) = \chi(\Sigma^x) + M = n(1-k) + M$.

(c) will also tell us the number of boundary components, $b$. Then

$2 - 2g - b = n(1-k) + M$
For instance, try \( n = 3, k = 3 \):

\[
\begin{align*}
    a_1 A_1 & \rightarrow y_1, \\
    b_1 B_1 & \rightarrow y_2, \\
    c_1 C_1 & \rightarrow y_3
\end{align*}
\]

Choose \( \sigma_a, \sigma_b, \sigma_c \in S_3 \), say \( \sigma_a = (123), \sigma_b = (12), \sigma_c = (23) \)

i.e. identify

\[
\begin{align*}
    a_1 &= A_2, \\
    b_1 &= B_2, \\
    c_1 &= C_2
\end{align*}
\]

Then after filling in the punctures, we have

\[
\begin{align*}
    \{ y_1 = z_1 = w_1, y_2 = z_2, w_2, y_3, z_3 = w_3 \} & \rightarrow \{ x_1, x_2, x_3 \} \\
    P & \rightarrow \Sigma
\end{align*}
\]

So, \( m = 5 \), and one can verify that \( b = 3 \), \( g = 0 \) and

\[
2 - 2g - 3 = 3(1-3) + 5 \Rightarrow g = 0 \quad \text{and} \quad \Sigma \cong \text{\#}
\]

Note: \( \Sigma \) has one branch point of index 3, two of index 2, and two of index 1.
A particularly useful family:

2-fold \( p: \Sigma \to D^2 \) with \( k \) branch points
in \( D^2 \) and all branch points in \( \Sigma \)
with index 2

We know \( \chi \in \mathbb{Z} \). In fact,

\[
b = \begin{cases} 
1 & \text{if } k \text{ odd} \\
2 & \text{if } k \text{ even}
\end{cases}
\]

Now \( 2 - 2g = \chi(\Sigma) + b = \begin{cases} 2(1-k)+k+1, & k \text{ odd} \\
2(1-k)+k+2, & k \text{ even}
\end{cases} \]

\[
-2g = 1-k
\]

\[
g = \frac{k-1}{2}
\]

\[
\begin{cases} 
k-1/2, & k \text{ odd} \\
k-2/2, & k \text{ even}
\end{cases}
\]

So, \( g = \begin{cases} 
k-1/2, & k \text{ odd} \\
k-2/2, & k \text{ even}
\end{cases} \)

i.e.,

\[
\begin{array}{c}
\circ \\
\downarrow \\
\circ \circ \circ \\
\cdots \cdots \\
\cdots \cdots \cdots 
\end{array}
\]
There is a sort of "tautological correspondence" that will help us understand maps \( u : \mathbb{D}^2 \to \text{Sym}^g(\Sigma) \). This will be a bijection:

\[
\begin{aligned}
\{ \text{holomorphic} \} \quad & \longleftrightarrow \quad \left\{ \begin{array}{l}
\text{diagrams} \\
\begin{array}{c}
S \\
\downarrow u_S \\
\mathbb{D}
\end{array}
\end{array} \right\} \\
\text{where} \\
\bullet S \text{ is Riemann surface} \\
\bullet u_S \text{ is } g\text{-fold branched cover} \\
\bullet u_D, u_S \text{ hol.}
\end{aligned}
\]

\[\leftarrow \right]: \text{Given } x \in \mathbb{D}, \text{ let } u_D^{-1}(x) = \{ y_1, \ldots, y_g \} \]
\[\text{where the } y_i \text{ are listed with multiplicity (based on index if } x \text{ is a branch point)}
\]

Then let \( u(x) := \{ u_S(y_1), u_S(y_2), \ldots, u_S(y_g) \} \in \text{Sym}^g(\Sigma) \).

\[\rightarrow \]: \text{Note that } \Delta \subset \text{Sym}^g(\Sigma) \text{ is a subvariety of codimension } 2. \text{ So, } \text{im}(D) \cap \Delta \text{ is a finite number of points. Let } D^x := D \setminus u^{-1}(\Delta).

\text{let } p : \Sigma \times \text{Sym}^{g-1}(\Sigma) \to \text{Sym}^g(\Sigma) \text{ be}
\]
\[p(x, \{ y_1, \ldots, y_{g-1} \}) = \{ x, y_1, \ldots, y_{g-1} \}. \text{ Then consider}
\]

\[
\begin{array}{ccc}
S^x & \longrightarrow & \Sigma \times \text{Sym}^{g-1}(\Sigma) \setminus p^{-1}(\Delta) \\
\downarrow & & \downarrow p \text{-} g\text{-fold covering} \\
D^x & \longrightarrow & \text{Sym}^g(\Sigma) \setminus \Delta
\end{array}
\]

Construct the pullback of } p \text{ via } u \text{ and call it } S^x. \text{ Then consider } u_D^x \text{ is } g\text{-fold covering.
There is a unique way to extend $u^x_D$ to $u_D : S \to D$ such that this diagram commutes:

$$
\begin{array}{ccc}
S & \xrightarrow{u_s} & \Sigma \times \text{Sym}^g(\Sigma) \\
\downarrow u_D & & \downarrow \rho \\
D & \xrightarrow{u} & \text{Sym}^g(\Sigma)
\end{array}
$$

Now let $u_\Sigma := \Pi \circ u_s$, where

$$
\Pi : \Sigma \times \text{Sym}^g(\Sigma) \to \Sigma
$$

is the projection onto the first factor.

(Won't prove $u_D$, $u_\Sigma$ holomorphic).

The idea here is to think of $D \xrightarrow{u} \text{Sym}^g(\Sigma)$ as a multivalued map into $\Sigma$. Branch points for $S \xrightarrow{u_D} D$ correspond to factors agreeing in $\text{Sym}^g(\Sigma)$, which in turn correspond to points "merging" in multivalued image in $\Sigma$.

E.g.

\begin{align*}
\includegraphics[width=0.5\textwidth]{example_diagram.png}
\end{align*}
Note: In the above examples (and all applications of this correspondence), we assume \( \mathcal{M} \) is a Whitney disk and obtain appropriate boundary conditions on \( u_\Sigma : S \to \Sigma \).
Proposition: $H_1(Sym^g(\Sigma)) \cong H_1(\Sigma)$ (\Sigma may have boundary)

Proof. Fix $x \in \Sigma$, and let

$$i: \Sigma \rightarrow Sym^g(\Sigma) \text{ be } i(y) = \{y, x, \ldots, x\}_{g-1}$$

We now construct an inverse $j$ for

$$i_*: H_1(\Sigma) \rightarrow H_1(Sym^g(\Sigma))$$

Let $\alpha \in H_1(Sym^g(\Sigma))$, and let $\gamma: S^1 \rightarrow Sym^g(\Sigma)$ have $[\gamma] = \alpha$. Generically $\gamma$ misses $\Delta$ (a codimension-2 subvariety of $Sym^g(\Sigma)$), and so lifts to $\tilde{\gamma}: S^1 \rightarrow \Sigma$, $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_g)$.

Define $j(\alpha) = [\gamma_1] + [\gamma_2] + \ldots + [\gamma_g] \in H_1(\Sigma)$.

Why is this well-defined? If two such $\gamma, \gamma'$ are homologous, then there is a cobordism $Z$ in $Sym^g(\Sigma)$ intersecting $\Delta$ in a finite number of points; a construction analogous to the "tautological correspondence" one gives a branched cover:

$$\tilde{Z} \rightarrow \Sigma \times Sym^g(\Sigma) \rightarrow \Sigma$$

$$\downarrow$$

$$Z$$

and the image of $\tilde{Z}$ in $\Sigma$ is a null-homology for $[\gamma] - [\gamma']$.

It can be verified that $j$ is an inverse for $i_*$. 

(Exercise)
Remarks

1. In fact, \( \pi_1(\text{Sym}^3(S)) \cong H_1(S) \) also.
2. A similar proof gives that when \( g > 2 \),
   \( \pi_2(\text{Sym}^3(S)) \cong H_2(\text{Sym}^3(S)) \cong H_2(S) \)
3. \( \pi_2(\text{Sym}(S)) = 0 \), \( \pi_2(\text{Sym}^2(S)) \cong \pi_1(\text{Sym}^2(S)) \)

Now we want to understand the structure of
\[ \pi_2(x,y) := \left\{ \text{homotopy classes of Whitney disks from } x \text{ to } y \right\} \]
and
\[ \hat{\pi}_2(x,y) := \left\{ \phi \in \pi_2(x,y) \mid \eta_z(\phi) = 0 \right\} \]
Note that \( \pi_2(x,x) \) (resp. \( \hat{\pi}_2(x,x) \)) acts freely, transitively on \( \pi_2(x,y) \) (resp. \( \hat{\pi}_2(x,y) \)).
So, should study \( \pi_2(x,x), \hat{\pi}_2(x,x) \).

Let \( \Omega(T_a, T_b) = \left\{ \gamma : [0,1] \rightarrow \text{Sym}^3(S) : \gamma(0) \in T_a, \gamma(1) \in T_b, \text{continuous} \right\} \)
and \( \tilde{\Omega}(T_a, T_b) = \left\{ \gamma : [0,1] \rightarrow \text{Sym}^3(S) : \gamma(0) \in T_a, \gamma(1) \in T_b, \text{continuous} \right\} \)

Observe:
1. \( (T_2(x,y) \neq \emptyset) \iff (x, y \text{ in same path}) \)
2. \( \pi_2(x,x) \cong \pi_1(\Omega(T_a, T_b)) \)
Proposition: $\Pi_0(\hat{\nabla}((\Pi_x, \Pi_\beta))) \cong H_1(Y)$ \,(g > 1)

$\Pi_1(\hat{\nabla}((\Pi_x, \Pi_\beta))) \cong H_1(Y)$

Pf: Consider the Serre fibration

$$\Omega \text{Sym}^g(\Sigma \setminus \Sigma) \rightarrow \hat{\nabla}((\Pi_x, \Pi_\beta)) \rightarrow \Pi_x \times \Pi_\beta$$

endpoint map

Also recall that

$\Pi_n(\Omega(M)) \cong \Pi_{n+1}(M)$ via

$$(D^n \rightarrow \text{Map}(\{0,1\}, M)) \leftrightarrow (D^n \times \{0,1\} \rightarrow M) \leftrightarrow (D^{n+1} \rightarrow M)$$

So, $\Pi_1(\Omega(\text{Sym}^g(\Sigma \setminus \Sigma))) \cong \Pi_2(\text{Sym}^g(\Sigma \setminus \Sigma)) = 0$ and $\Pi_0(\Omega(\text{Sym}^g(\Sigma \setminus \Sigma)) \cong \Pi_1(\text{Sym}^g(\Sigma \setminus \Sigma)) = H_1(\Sigma)$

So, have S.E.S. (coming from L.E.S. of fibration):

$$0 \rightarrow \Pi_1(\hat{\nabla}) \rightarrow \Pi_1(\Pi_x \times \Pi_\beta) \xrightarrow{q} H_1(\Sigma) \rightarrow \Pi_0(\hat{\nabla}) \rightarrow 0$$

Now $\Pi_0(\hat{\nabla}) = \text{img} \cong H_1(\Sigma) / \text{im} f \cong H_1(Y)$

$\Pi_1(\text{Sym}^g(\Sigma \setminus \Sigma)) \cap \text{im} f \cong \langle [\xi_1], ..., [\xi_g], [\beta_1], ..., [\beta_g] \rangle \subset H_1(\Sigma)$

We proceed by studying Mayer-Vietoris (for reduced cohomology) for $Y = U_\alpha \cup_\Sigma U_\beta$. 
$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$

$0 \rightarrow H^0(\Sigma) \rightarrow H^0(U_\alpha) \oplus H^0(U_\beta) \rightarrow H^0(\Sigma) \rightarrow \ldots$

$\hookrightarrow H^1(\gamma) \hookrightarrow H^1(U_\alpha) \oplus H^1(U_\beta) \rightarrow H^1(\Sigma) \rightarrow \ldots$

Note: Under the identification $\pi_1(Sym^2(\Sigma^2)) \cong H^1(\Sigma)$, $\pi_1(T_{\alpha}) \cong H^1(U_\alpha)$ and $\pi_1(T_{\beta}) \cong H^1(U_\beta)$. So,

$H^1(U_\alpha) \oplus H^1(U_\beta) \overset{h}{\rightarrow} H^1(\gamma) \overset{\cong}{\Rightarrow} \pi_1(Sym^2(\Sigma^2))$

As a result, we obtain the short exact sequence

$0 \rightarrow \pi_1(\Sigma) \rightarrow H^1(\gamma) \rightarrow 0$. $\square$

---

In summary, we'll partition $\pi_1(T_{\alpha} \cap T_{\beta})$ into $H^1(\gamma)$-equivalence classes. If $x, y$ are in the same class, then $\hat{\pi}_2(x, y)$ is non-empty and is an affine space over $\hat{\pi}_2(x, x) \cong H^1(\gamma)$.

**Rmk:** When $g = 1$, $\hat{\pi}_2(x, x) \rightarrow H^1(\gamma)$, and this is still enough for our later purposes.
So, we should be thinking of \( H_1(y) = \mathbb{Z}^{b_1} \oplus \text{(Torsion)} \) and \( H'(y) \cong \mathbb{Z}^{b_1} \).

**The landscape:**

Recall:

1. \( T_\alpha, T_\beta \subset \text{Sym}^9(\Sigma \times \Sigma) \) cpt.
2. \( \text{Sym}^9(\Sigma \times \Sigma) \) convex at infinity
3. \( \text{Would like to show that } E(\ker(\mu : \hat{H}_2(x,x) \to \mathbb{Z})) \) is bounded (\( \Rightarrow \) Gromov cptness applies, \( d^2 = 0 \))

**Note:** If \( b_1 = 0 \), then \( \hat{H}_2(x,x) = 0 \Rightarrow \mu \) injective.

So, we can define \( \hat{H}_F \) for \( \mathbb{Q} - H_* \)-spheres.
We’d like to study $\phi \in \Pi_2^0(x,y)$ by studying domains in $\Sigma$.

Let $R_1,\ldots,R_n$ be closures of the components of $\Sigma \setminus (\alpha \cup \beta)$, and choose $z_i \in R_i^0$ (for each $i$).

**Def:** let $u$ be a Whitney disk from $x$ to $y$. The domain of $u$ is the two-chain

$$D(u) := \sum_{i=1}^n n_{z_i}(u) R_i$$

**Note:**
1. $D(u)$ only depends on $[u] \in \Pi_2^0(x,y)$
2. $D(u) = \text{Im}(U_{z_i})$ from tautological correspondence
3. When $u$ is holomorphic, $n_{z_i} \geq 0$ for all $i$.  

\[ \hat{\mathcal{C}}F(S^3) = \langle x \rangle, \quad \varepsilon = 0 \]

\[ \Rightarrow \hat{\mathcal{H}}F(S^3) \cong \mathbb{Z} \]

\[ \hat{\mathcal{C}}F(L(p,1)) \cong \langle x_1, x_2, x_3 \rangle, \]

\[ \varepsilon = 0 \]

\[ \Rightarrow \hat{\mathcal{H}}F(L(p,1)) = \mathbb{Z}^p \]

\[ = \bigoplus \mathbb{Z} \quad \text{if } H_1(L(p,1)) \]
Def: Let \( x, y \in \Pi_a \cap \Pi_B \), choose a pair of paths from \( x \) to \( y \): \( a : [0,1] \to \Pi_a, b : [0,1] \to \Pi_B \).

Let \( e(x, y) \) be the image in \( H_1 (\Sigma ; \mathbb{Z}) \) of \( [a-b] \) under
\[
\frac{H_1(\text{Sym}^g(\Sigma))}{\pi_1(\Sigma) \otimes H_1(\Pi_B)} = H_1(\Sigma) \cong H_1(\Sigma)
\]

\( (e(x, y) \) doesn't depend on the paths chosen).

Rmk: \( \phi \in \pi_2(x, y) \) \( ([u] = \phi) \Rightarrow \)
\[
[u(\text{red})]_{\pi_1(\text{Sym}^g(\Sigma))} = 0 \Rightarrow e(x, y) = 0.
\]

One can study \( e(x, y) \) in \( \Sigma \). Let \( x, y \in \Pi_a \cap \Pi_B \).

Then choose a family of arcs in the \( \alpha_i \) connecting \( x \) to \( y \) and a family of arcs in the \( \beta_i \) connecting \( y \) to \( x \). The sum of these 1-chains is a 1-cycle representing \( e(x, y) \).

ex: \( L(p, g) \): here's the Heegaard diagram for \( p = 3, g = 2 \).

We claim that all of \( e(a, b), e(a, c), \) and \( e(b, c) \) are nonzero.

For instance, the loop exhibits that
\[
e(a, c) = 2[m] \neq 0
\]
Suppose that \( x, y \in \Pi_0 \cap \Pi_\beta \), where
\[
X = \{x_1, \ldots, x_g\}, \quad Y = \{y_1, \ldots, y_g\}, \text{ and }
\]
- \( x_i \in \alpha_i \cap \beta_i \)
- \( y_i \in \alpha_i \cap \beta_{\sigma^{-1}(i)} \) \( (\sigma \in S_g \text{ is some permutation}) \)

**Def:**

(a) A domain connecting \( x \) to \( y \) is a formal linear combination \( A = \sum_{i=1}^{N} a_i R_i \) \( (a_i \in \mathbb{Z}) \) s.t.
- \( \partial A |_{d_i} \) is a 1-chain with boundary \( y_i - x_i \)
- \( \partial A |_{R_i} \) is a 1-chain with boundary \( x_i - y_{\sigma(i)} \)

We denote the set of these as \( D(x, y) \).

(b) Given \( \phi \in \Pi_2(x, y) \), the domain of \( \phi \) is
\[
D(\phi) = \sum_{i=1}^{N} n_{\phi_i}(\phi) R_i
\]

Note: \( D(\phi) \in D(x, y) \)

**Proposition:** The map \( D : \Pi_2(x, y) \rightarrow D(x, y) \) is a
\[
\begin{cases}
\text{bijection if } g > 2 \\
\text{surjection if } g = 2 \\
\text{injection if } g = 1
\end{cases}
\]

**Rmk:** (a) When \( g = 2 \), replace \( \Pi_2(x, y) \) with its quotient by \( (\phi_1 \sim \phi_2 \Rightarrow D(\phi_1) = D(\phi_2)) \)
(b) \( g = 1 \) isn't a big deal - maps \( u : D \rightarrow \Sigma \) are easy to study.
Recall: We also defined an element 
\[ e(x,y) \in H_1(Y), \] and saw that 
\[ e(x,y) \neq 0 \implies \pi_2(x,y) = \emptyset. \]

**Proposition:** \[ e(x,y) = 0 \implies D(x,y) \neq \emptyset \left( \implies \pi_2(x,y) \neq \emptyset \right) \]

**Pf:** We computed \( e(x,y) \) by selecting \( \alpha : [0,1] \to \pi_\alpha \) from \( x \) to \( y \) and \( \beta : [0,1] \to \pi_\beta \) from \( y \) to \( x \), and then setting \( [\alpha + \beta]_{H_1(Y)} = e(x,y) \) via

the identification

\[
\frac{\pi_1(\text{Sym}^g(\Sigma))}{\pi_1(\pi_\alpha) \oplus \pi_1(\pi_\beta)} \cong \frac{H_1(\Sigma)}{\langle [\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g] \rangle} \cong H_1(Y)
\]

So, view \( \alpha + \beta \) as a 2-chain in \( H_1(\Sigma) \).

\[ e(x,y) = 0 \implies [\alpha + \beta] \in \text{Span}(\langle [\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g] \rangle)
\]

Then there is some \( \eta \) which is a linear combination of \( \alpha_i \)'s and \( \beta_i \)'s s.t. \( [\alpha + \beta]_{H_1(Y)} = [\alpha + \beta - \eta]_{H_1(Y)} \)

and \( [\alpha + \beta - \eta]_{H_1(Y)} = 0 \).

So, \( \gamma = \alpha + \beta - \eta \) bounds a 2-chain \( X = \sum_{i=1}^n a_i R_i \), and

\[ \partial X = \sum_i a_i \partial R_i \]

are such that \( X \in D(x,y) \).
Rmk: By adding some multiple of the entire surface to \( \mathbf{A} \), if necessary, we can assume that \( n_z(\mathbf{A}) = 0 \).

One can recover the coefficients \( a_i \) in the expression \( \mathbf{A} = \sum_{i=1}^{N} a_i \mathbf{R}_i \) via the following procedure:

1. Let \( \Gamma \) be the graph with vertices \( \{ z_i \}_{i=1}^{N} \) and edges \( (z_i, z_j) \) when \( \mathbf{R}_i \) borders \( \mathbf{R}_j \) (i.e., the "dual graph" of the 1-skeleton of the cellular decomposition specified by \( \mathbf{R}_1, \ldots, \mathbf{R}_N \)).

2. Choose any spanning tree \( T \) of \( \Gamma \).

3. Label the vertices of \( T \) by first labelling \( z_i = z_i \) with 0 and then inducing labels on all others via the following rule:

   If \( (z_i, z_j) \) is an edge and \( z_i \) already has label \( \alpha \in \mathbb{Z} \), then give \( z_j \) the label \( \alpha + n \), where \( n \in \mathbb{Z} \) is the local coefficient of the boundary arc \( \lambda : z_i \rightarrow z_j \) in the 1-chain \( \mathbf{A} \).
Since $T$ is a spanning tree, this procedure will successfully label all vertices.

\textbf{Rmk.}

1. The label on the vertex $x_i$ is exactly the coefficient $a_i$ of $R_i$ in $A$.

2. The domain $A$ is known to exist; we're simply recovering its local coefficients by imposing the necessary condition that if $\lambda$ has coefficient $n \in \mathbb{Z}$ in $\partial A = \gamma$ and $\lambda$ separates regions with coefficients $a_i, b \in \mathbb{Z}$ in $A$ as in $a \downarrow b$, then $n = b - a$, and so $b = a + n$.

(\textit{so, the $a_i$ we find don't depend on the tree, i.e., $T$})

\textbf{Ex.}

\(g \geq 3\) Assume that locally we have

Then we may use $a + b = \text{to compute } \varepsilon(x, y) = 0$. Notice $[a + b]_{H_1(\Sigma)} = 0$, so we can use $Y = a + b$, and $Y = \partial A$ where $A$ has coefficients:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

(verify this)

\[
\begin{array}{c}
\Delta \quad \Delta \quad \Delta \\
\Delta \quad \Delta \quad \Delta \\
\Delta \quad \Delta \quad \Delta
\end{array}
\]

"o" are components of $X$

"A" are components of $Y$
Let $m_1, l_1$ be:

Exercise: $\alpha_1 = l_1$, $\beta_1 = l_1 - l_2 - 2m_1$, $\beta_2 = m_1 + m_2$ (as 1-cycles)

So, $H_1(Y) \cong \langle m_1, m_2, l_1, l_2 \rangle / \langle l_1, l_2, l_1 - l_2 - 2m_1, m_1 + m_2 \rangle \cong \mathbb{Z} / \langle m_1 \rangle$

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be as labelled.

Then to compute $\varepsilon(x, y)$, one might use

$$a + b = \begin{array}{c}
\text{i.e.} \\
a + b = 2m_2
\end{array}$$

Now $[a + b]_{H_1(Y)} \neq 0$, but

$[a + b + \alpha_2 - \alpha_1 + 2\beta_2 + \beta_1]_{H_1(Y)} = 0$ and $[a + b + \eta]_{H_1(Y)} = [a + b] = 0$.

We use the algorithm to find some $\Delta$ with

$$\partial \Delta = a + b + \eta.$$ The result has local coefficients as on the following page...
\[ \text{Spin}^c \text{-structures: } (\gamma^3 \text{- closed, oriented, connected}) \]

**Def.** Let \( \gamma, \nu \) be nowhere-vanishing v.f. on \( \gamma \).

\( \gamma \) is homologous to \( \nu \) if there is a ball \( B^3 \subset \gamma \)

\( \gamma/\gamma \backslash B \) homotopic to \( \nu/\nu \backslash B \).

\[ \text{Spin}^c(\gamma) = \left\{ \text{homology classes of nowhere-vanishing v.f. on } \gamma \right\} \]

There is another formulation:

\( \text{SO}(3) \cong \text{SU}(2)/\mathbb{U}(1) = \text{SU}(2)/\text{U}(1) \).

This gives a principal bundle

\[ \text{U}(1) = S^1 \longrightarrow \text{U}(2) \quad \text{Then } \text{Spin}^c(3) = \text{U}(2) \text{ and } \]

\[ \text{Spin}^c(3) = (\text{U}(1) \times \text{Spin}(3))/\{ \pm 1 \} \]

There is an \( \text{SO}(3) \)-bundle

\[ \text{SO}(3) \longrightarrow F \gamma \quad \text{y} \]

**Def.** A \( \text{Spin}^c \)-structure on \( \gamma \) is a lift of \( y \) to

a principal \( \text{Spin}^c(3) \)-bundle.

(i.e. a pair \((S, \alpha)\), where

\( S = \text{a principal } \text{Spin}^c(3) \text{-bundle over } \gamma \)

\( \alpha: \gamma/\gamma \backslash U(1) \longrightarrow F \gamma \), an \( \text{SO}(3) \)-bundle isomorphism)

We'll use the first formulation.

Fix a trivialization \( \mathcal{T} \) of \( T\gamma \). \( \mathcal{T} \) induces

\[ \left\{ \text{nowhere-vanishing v.f, v on } \gamma \right\} \quad \longleftrightarrow \quad \left\{ \text{maps } f: \gamma \longrightarrow \mathbb{R}^3 \setminus 0 \right\} \]
Now let $\mu$ be the positive generator of $H^2(S^1 \times 0) \cong H^2(S^2) \cong \mathbb{Z}$.

**Def:** Let $S^\tau(\nu) := f^*_{\nu}(\mu) \in H^2(Y; \mathbb{Z})$

**Exercises:**

1. $S^\tau : \text{Spin}^c(Y) \to H^2(Y; \mathbb{Z})$ is a bijection.
2. $S(\nu_1, \nu_2) := S^\tau(\nu_1) - S^\tau(\nu_2) \in H^2(Y; \mathbb{Z})$
   doesn't depend on choice of $\tau$.

So, we get an action of $H^2(Y; \mathbb{Z})$ on $\text{Spin}^c(Y)$:

$a \in H^2(Y), \nu \in \text{Spin}^c(Y)$

Then $a + \nu$ is the element of $\text{Spin}^c(Y)$ with $S(a + \nu, \nu) = a$

**Rmk:** We'll often write $\nu_1 - \nu_2$ to denote $S(\nu_1, \nu_2)$

We can now define a map $S_\tau : \prod_a \cap \prod_b \to \text{Spin}^c(Y)$.

An alternative view of $H$-diagrams:

**Recall:**

**Def:** A Morse function $f : Y \to \mathbb{R}$ is one with no non-degenerate critical points.

($p \in \text{Crit}(f)$ is non-degenerate if $\text{Hess}_p(f)$ is non-singular)

At $p$, $f$ looks locally like

$f(p) - (x_1^2 + \ldots + x_i^2) + (x_{i+1}^2 + \ldots + x_d^2)$

($i$ is the **index** of $p$)

**Def:** $f$ (Morse) is **self-indexing** if $f(p) = \text{index}(p)$, $p \in \text{Crit}(f)$
Thm (Morse): A self-indexing Morse function $f: Y^d \to \mathbb{R}$ always exists, and can be chosen with unique cp. of index 0, d.

Note: $f$ gives a handle decomposition of $Y^d$

ex: $Y^2 = \begin{array}{c} 0 \end{array} \xrightarrow{f} \begin{array}{c} 1 \end{array} \xrightarrow{3} \begin{array}{c} 2 \end{array} \xrightarrow{2} \begin{array}{c} 1 \end{array} \xrightarrow{0} \begin{array}{c} 3 \end{array}$

(f = height)

[This $f$ is not self-indexing]

In general, an index-$k$ cp. corresponds to a $k$-handle.

So, we can find $f: Y^3 \to \mathbb{R}$ (Morse) with

- 1 index-0 cp.
- 9 index-1 cp. $x_1, \ldots, x_9 \in f^{-1}(1)$
- 9 index-2 cp. $y_1, \ldots, y_9 \in f^{-1}(2)$
- 1 index-3 cp.

Now $f$ determines a Heegaard diagram for $Y$:
In this diagram, label the index-1 critical points $p_1, \ldots, p_g$ and the index-2 ones $q_1, \ldots, q_g$.

Rmk: Near some $p_i$, there is a chart $\varphi : N_{p_i} \to \mathbb{R}^3$ s.t. locally, it looks like

$$f(\varphi^{-1}(x, y, z)) = f(p_i) + x^2 + y^2 - z^2$$

So, $-\nabla f = -2<x, y, -z>$, i.e.

We can see that the set of points (locally) which flow to the origin under $-\nabla f$ flow is disk in the $(xy)$-plane.

Since there are no critical values in $f^{-1}([1, 3/2])$, the set of points in $f^{-1}([1, 3/2])$ which flow to $p_i$ under $-\nabla f$ is a disk with boundary in $f^{-1}(3/2)$.

In the preceding diagram:

- $\Sigma := f^{-1}(3/2)$ is a genus-$g$ surface.
- $U_\alpha := f^{-1}([0, 3/2])$ and $U_\beta := f^{-1}([3/2, 3])$ are handlebodies.
- $\alpha_i := \{\text{points in } \Sigma \text{ flowing to } p_i \text{ under } -\nabla f\}$ is a s.c.c. bounding a disk in $U_\alpha$.
- $\beta_i := \{\text{ } q_i \text{ } + \nabla f\}$ is a Heegaard splitting.
- $z \in \Sigma \cup (\alpha \cup \beta)$, which generates a flowline connecting the critical pts. of index 0 and 3.
- $Y = U_\alpha \cup U_\beta$ is a Heegaard splitting.
Fact: Any diagram arises from such a Morse function $f$.
(We say $f$ is compatible with $H$)

We can use the function $f$ to construct $S_z$.

1. $w = (w_1, ..., w_g) \in T_a \cap T_p$ determines a g-tuple of trajectories for $\nabla f$ connecting $c_p$, of index 1 and 2:

\[
(\forall w_i \in \alpha_i \cap \beta_i, i \in g) \\
\sigma \in S_g
\]

2. Delete tubular nhbd $B_i^3$ of $(g+1)$-trajectories, $\nabla f$ is nonvanishing in $Y \setminus (\bigcup_{i=1}^{g+1} B_i)$.

3. Can extend $\nabla f/\!\!/Y \setminus B$ ($B = \bigcup_{i=1}^{g+1} B_i$) to some nonvanishing v.f. on $Y^3$.

This v.f. agrees with $\nabla f$ outside of $B$, so determines a class $S_z(w) \in \text{Spin}^c(Y)$. 
Proposition: (Won't prove here)

(a) \( s_z(y) - s_z(x) = \text{P.D.}(\epsilon(x,y)) \)

(\( \Rightarrow s_z(y) = s_z(x) \iff \pi_2(x,y) = \emptyset \))

(b) If \( z_1, z_2 \) can be connected as so:

Then \( s_{z_1}(x) - s_{z_2}(x) = \pm \text{P.D.}[\alpha_i^*] \),

where \( \alpha_i^* \) is the "dual curve" to \( \alpha_i \),

i.e. \( \alpha_i^* \in \Sigma \) with \( \alpha_j^* \cdot \alpha_j^* = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ else} \end{cases} \)

So, we can use \( s_z \) to partition \( \pi_2(x,y) \) into \( \text{Spin}^c(Y) \)-equivalence classes, i.e. \( \hat{CF}(\mathcal{H}) = \bigoplus_{s \in \text{Spin}^c(Y)} \hat{CF}(\mathcal{H}, s) \).

Def: (a) Given \( s \in \text{Spin}^c(Y) \) (with \( [s] = s \)),

\( \bar{s} = [-v] \) is the conjugate of \( s \).

(b) There is a map

\( c_1: \text{Spin}^c(Y) \to H^2(Y) \) given by

\( c_1(s) = s - \bar{s} \) (\( \equiv \delta(s, \bar{s}) \)), the "first Chern class of \( s \)."

Rmk: 0 If \( s = [v] \), \( c_1(s) \) coincides with the first Chern class of the \( L \)-line bundle over \( Y \) given by \( L \).

2. \( \frac{c_1}{\alpha}: \text{Spin}^c(Y) \to H^2(Y) \) (this is canonical)
We'd like to have that $E(\ker \alpha; \pi_2(x,x) \to \mathbb{Z})$ to be bounded for $b_i > 0$.

This would lead to an a priori energy bound $C(x,y)$ on holomorphic index-1 Whitney disks from $x$ to $y$, which would in turn imply that

$$ \bigcup_{\phi \in \pi_2(x,y), M(\phi) = 1} \{ M(\phi) \}$$

is cpt.

To this end, we impose a technical condition on Heegaard diagrams: First, some definitions.

**Def:**

1. A **periodic domain** is a two-chain $P = \sum a_i R_i$ such that $\partial P$ is a union of $\alpha$ and $\beta$ circles, (i.e. $\partial(P) = \emptyset$) and $n_2(P) = 0$.

2. A class $\phi \in \pi_2(x,x)$ is periodic if $n_2(\phi) = 0$ ($\Rightarrow D(\phi)$ is a periodic domain)

   - the periodic classes are exactly $\hat{\pi}_2(x,x) \subset \pi_2(x,x)$

**E.g.**

1. This is the domain of some $\phi \in \pi_2(x,x)$ (and some $\gamma \in \pi_2(y,y)$)

2. This is a periodic domain, but isn't the domain of a periodic class.
\[ \Pi_x(z) := \left\{ \text{periodic domains connecting } x \text{ to } x \right\} \]

Clearly \( H'_1(Y) \cong H_2(x, x) \cong \Pi_x(z) \)
\( \phi \mapsto D(\phi) \)

*E.g.*
1. \( S^2 \times S^1 : H'_1(Y) \cong \mathbb{Z} \)
   \[ \mathbb{Z} \cong \mathbb{N} \]
   \[ \xrightarrow{\alpha} - \phi \]
   \[ \subset \Pi_x(z) \]

2. \( S^3 : H'_1(Y) = 0 \)
   No periodic domains from \( x \) to \( x \) except \( 0 = D(\text{constant class at } x) \)

**Def:** A pointed Heegaard diagram is *weakly admissible* if there is a signed area form on \( \Sigma \) giving all domains of periodic classes area = 0.

**Remarks:**
1. If no periodic domains, then weak admissibility is automatic.
2. All periodic domains have positive and negative coefficients \( \Rightarrow \) weak admissibility.
3. Weak admissibility avoids having to use \( E(\ker \mu) \text{ bdd.} \)
Lemma: (Oz−Sz): Let (Σ, α, β, z) be weakly admissible. Then for each \( x, y \in \Pi_0 \cap \Pi_1 \), there are only finitely-many \( \phi \in \pi_2(x, y) \) with \( \mu(\phi) = j, \ n_x(\phi) = k, \) and \( D(\phi) \neq 0 \).

**Pf:** Fix \( \psi \in \pi_2(x, y) \) with \( \mu(\psi) = j, \ n_x(\psi) = k \). Then given \( \phi \in \pi_2(x, y) \), there is some periodic class \( \eta \) with

\[
D(\phi) = D(\psi) + D(\eta)
\]

So, \( D(\eta) \geq -D(\psi) \) if \( D(\phi) \geq 0 \).

But then the set \( \{ \phi \in \pi_2(\Sigma) \mid P \geq -D(\psi) \} \) is finite by admissibility (since \( P \) must have some negative coefficients).

So, weak admissibility guarantees that the sum in

\[
dx = \sum_{\gamma \in \Pi_0 \cap \Pi_1} \sum_{\phi \in \pi_2(x, y) \atop \mu(\phi) = 1} \#(M(\phi)/R) \gamma
\]

is finite since

- \( M(\phi)/R \) compact and 0-dimensional
- \( \phi \) has holomorphic representatives \( \Rightarrow D(\phi) \geq 0 \).
Proposition: (won't prove):

1. Every pointed Heegaard diagram can be isotoped to be weakly admissible.

   ![Diagram]

   e.g. \( \alpha \)

2. Every two weakly admissible diagrams for \( Y \) can be related by a sequence of Heegaard moves through weakly admissible diagrams.

So, for any \( Y^3 \), we can define \( \hat{HF}(Y) \).

Gradings (relevant when \( b_1(Y) > 0 \))

**Def:** Define \( M_s : H'(Y) \rightarrow \mathbb{Z} \) by

(i) \( M_s(h) = (c_i(s) \cup h)[Y] \) and

(ii) \( \delta(s) = \gcd \left\{ M_s(h) \mid h \in H'(Y) \right\} \)

**Fact:** If \( \phi \in \hat{\pi}_2(x,x) \) with \( s_z(x) = s \), then

\[
M(\phi) = M_s(h(\phi))
\]

\( (h : \hat{\pi}_2(x,x) \rightarrow H'(Y)) \)

So, the set \( \left\{ x \in \mathbb{T}_a \cap \mathbb{T}_b \mid s_z(x) = s^3 \subset \mathbb{T}_a \cap \mathbb{T}_b \right\} \) has a well-defined \( \left( \mathbb{Z} / \delta(s) \mathbb{Z} \right) \)-grading given by

\[
gr(x,y) \equiv M(\phi) \pmod{\delta(s)}, \quad \phi \in \hat{\pi}_2(x,y).
\]
e.g.

1. \( S^2 \times S^1 \)

\[ \begin{align*}
\alpha & \qquad \text{via orientation} \\
\delta x = y - y &= 0 \\
\delta y &= 0 \\
\end{align*} \]

(homology spaces)

\[ \text{So, } \hat{\text{HF}}(S^2 \times S^1) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)} \]

But which \( \hat{\text{HF}}(S^2 \times S^1, s) \) is non-zero? \( (\text{Spin}^c(S^2 \times S^1) \cong \mathbb{Z}) \)

\( \hat{\text{HF}}(S^2 \times S^1, s) \) is relatively \( \mathbb{Z}/2s \)-graded

\( \pi_2(x,x) \cong \mathbb{Z} \) is generated by \( D_1 - D_2 \), where \( D_i \) are as in diagram above.

\[ m(D_1 - D_2) = m(D_1) - m(D_2) = 1 - 1 = 0. \Rightarrow s = 0 \]

So,

\[ \hat{\text{HF}}(S^2 \times S^1, s) = \begin{cases} 
\mathbb{Z}, & \text{else} \\
0, & \text{else} 
\end{cases} \]

2.

\[ \begin{align*}
\alpha & = d-d + b-b = 0 \\
\delta b &= c = c + d \\
\delta c &= 0 \\
\end{align*} \]

So, we see

\[ C^F(H) = \left( \begin{array}{c}
\mathbb{Z} \\
\oplus \\
\mathbb{Z} \\
\end{array} \rightarrow \mathbb{Z} \right) \Rightarrow \hat{\text{HF}}(H) = (\mathbb{Z} \oplus \mathbb{Z} \oplus 0) \]

\( (a, b, c, d) \text{ all correspond to } s = 0 \)
Maslov index and domains:

Goal: Given $D(\phi)$, we want to compute $\mu(\phi)$.

Def: Let $D$ be a Riemann surface with boundary and corners on the boundary. Choose a metric on $D$ s.t.
- $\partial D$ is geodesic
- Corners are all right angles ($\square$ or $\bigtriangleup$)

Then the Euler measure of $D$ is

$$e(D) := \frac{1}{2\pi} \int_{D} K dA \quad \text{(curvature)}$$

Note: $e(D_1 \cup D_2) = e(D_1) + e(D_2)$

ex: ① $e(\Sigma_g) = 2 - 2g$ (Gauss–Bonnet theorem)

② $e(\bigtriangleup) = \frac{1}{4} e(\mathcal{O}) = \frac{1}{2}$

③ $e(\square) = \frac{1}{4} e(\square) = 0$

④ Gauss–Bonnet theorem:

$$\int_{D} K dA + \int_{\partial D} K d\sigma = 2\pi \chi(D)$$

Note: $2\pi \mu(D)$
Note: If $\partial D$ piecewise-smooth, then $\sum_{\partial D} \kappa$ is
\[ \sum_{i=1}^{m} \kappa_{i} + \sum_{i=1}^{m} \text{(interior angle at } i^{th} \text{ corner)} \]

So, in our case,
\[ e(D) = \chi(D) - \frac{1}{2\pi} \left( \sum_{i=1}^{m} \text{(i}^{th} \text{, interior angle)} \right) \]

So, if $D_{m,k}$ is an $(m+k)$-gon s.t. $\partial D_{m,k}$ has
$m$ acute corners and $k$ obtuse,
\[ e(D_{m,k}) = 1 - \frac{1}{2\pi} \left( m \left( \frac{\pi}{2} \right) + k \left( -\frac{\pi}{2} \right) \right) \]
\[ = 1 - \frac{m-k}{2} \]

If $D_{m,k}^{g,b}$ is genus-$g$ surface with $b$ boundary components
and $\partial D$ has a total of $m$ acute and $k$ obtuse angles,
\[ e(D_{m,k}^{g,b}) = \chi(D_{m,k}^{g,b}) - \frac{m-k}{2} = 2 - 2g - b - \frac{m-k}{2} \]

\[ \text{ex:} \quad D = \begin{array}{c}
\text{Diagram of a domain with marked angles and boundary components.}
\end{array} \]
\[ e(D) = 2 - 4 - 2 - \frac{10}{4} = -4 - \frac{5}{2} = -\frac{13}{2} \]

Now, extend $e$ additively to all domains.
Vertex multiplicity: Let $D$ be a domain from $x$ to $y$, $x, y \in \Pi_\alpha \cap \Pi_\beta$. Look at some $x_i \in \alpha_i \cap \beta_i \subset \Sigma$:

\[
A_{x_i} + B_{x_i} + C_{x_i} + D_{x_i} = 4 \\
\text{Let } n_{x_i}(D) := \frac{A_{x_i} + B_{x_i} + C_{x_i} + D_{x_i}}{4} \\
\text{and } n_x(D) := \sum_{i=1}^{g} n_{x_i}(D)
\]

$A_{x_i}, B_{x_i}, C_{x_i}, D_{x_i}$ are local multiplicities of regions in $D$.

\[\begin{align*}
\text{ex:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\circ\quad x
\end{array}
\end{array}
\end{array} & n_x(D) = n_y(D) = \frac{1}{4}
\end{align*}\]

\[\begin{align*}
\text{ex:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\circ\quad x
\end{array}
\end{array}
\end{array} & n_{x_1} = n_{x_2} = \frac{1}{4} \implies n_x = n_y = \frac{1}{2}
\end{align*}\]

\[\begin{align*}
\text{ex:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\circ\quad x
\end{array}
\end{array}
\end{array} & n_x = n_y = \frac{3}{4} \implies M(\phi) = -\frac{1}{2} + \frac{3}{2} = 1
\end{align*}\]

Thm (Lipschitz):
Let $\phi \in \Pi_2(x, y)$ and let $D = D(\phi)$. Then

\[M(\phi) = e(D) + n_x(D) + n_y(D)\]

\[\begin{align*}
\text{ex:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\circ\quad x
\end{array}
\end{array}
\end{array} & M(\phi) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1
\end{align*}\]

\[\begin{align*}
\text{ex:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\circ\quad x
\end{array}
\end{array}
\end{array} & M(\phi) = \frac{1}{2} + \frac{1}{2} = 1
\end{align*}\]

\[\begin{align*}
\text{ex:} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\circ\quad x
\end{array}
\end{array}
\end{array} & M(\phi) = \frac{1}{2} + \frac{1}{2} = 1
\end{align*}\]
So, we can now use Lipschitz's formula to compute $M(\phi)$ for any $\phi \in \pi_2(x,y)$:

1. Compute $e(D(\phi))$ (decomposing if necessary)
2. Compute $n_x(D(\phi)), n_y(D(\phi))$ using coefficients near $x_i, y_i$.

Now we want to verify that $\hat{HF}$ only depends on the 3-manifold $Y$.

(Actually, that $HF(Y,s)$ only depends on $(Y,s)$)

---

**Isotopy:** It's clear that there's a chain complex isomorphism if the isotopy preserves intersections. We allow other "controlled" isotopies.

Let $(M,w)$ be symplectic, and let $H_t : M \to \mathbb{R}$ ($t \in [0,1]$) be a smooth family of smooth functions. Since $w$ is non-degenerate, there is a unique family of vector fields $X_t$ with $i_{X_t}w = dH_t$.

**Def:** $X_t$ is called the time-dependent Hamiltonian vector field for $H_t$.

**Def:** Given $X_t$, there is a unique smooth family of diffeomorphisms $\phi_t : M \to M$ with

$$\frac{d}{dt} \phi_t = X_{H_t} \circ \phi_t, \quad \phi_0 = \text{id}_M.$$ 

$\phi_t$ is called the time-dependent Hamiltonian flow.
The resulting diffeomorphism $\phi_t : M \to M$ is called a Hamiltonian transformation of $M$.

Exercise: $\phi_t^* \omega = \omega$, $\forall t \in [0,1]$.

Thm: Let $\Psi$ be a Hamiltonian transformation.

Then $\text{HF}(L_0, L_1) \cong \text{HF}(L_0, \Psi(L_1)) \cong \text{HF}(\Psi(L_0), L_1)$.

(we won't prove this)

As long as we preserve area, then an isotopy can be achieved as a Hamiltonian transformation:

\[
\begin{array}{c}
\alpha \\
\beta
\end{array} \quad \rightarrow \quad \begin{array}{c}
\alpha' \\
\beta'
\end{array}
\]

2) Stabilization:

$H = (\Sigma, \alpha, \beta, z) \rightarrow H^+ = (\Sigma^+ \alpha^+, \beta^+, z^+)$

There's an obvious bijection $\Pi_{\alpha^+} \cap \Pi_{\beta^+} \leftrightarrow \Pi_{\alpha} \cap \Pi_{\beta}$

given by $x = \{x_1, \ldots, x_g, z\} \rightarrow x^+ = \{x_1, \ldots, x_g, \alpha^+\}$

There's an obvious identification $\hat{\Pi}_2(x, y) \leftrightarrow \hat{\Pi}_2(x^+, y^+)$

as domains avoid $z$ (and $\gamma$ supports no classes).

Furthermore, for $\phi \in \hat{\Pi}_2(x, y)$, $M(\phi)$ and $M(\phi^+)$ are homeomorphic via

$u \mapsto u.x$ (constant map).
(3) Handleslides:

We first discuss triangles:
Let $L_0, L_1, L_2 \in (M, \omega)$ be Lagrangians (with all technical assumptions to avoid bubbles).

Let $x \in L_0 \cap L_1$, $y \in L_1 \cap L_2$, $z \in L_0 \cap L_2$.

\[ \Pi_2(x, y, w) = \left\{ \text{homotopy classes of maps } u : D \to M, \right\} \]

where

\[ D = \begin{array}{c}
\text{v}_0 \\
\text{v}_2 \\
\text{v}_1 \\
\end{array} \]

\[ \begin{array}{c}
u(v_2) = x \\
u(v_0) = y \\
u(v_1) = w \\
\end{array} \]

\[ u(e_1) \in L_0 \\
u(e_2) \in L_1 \\
u(e_2) \in L_2 \]

(these maps are called Whitney triangles)

*Since a holomorphic map is determined by 3 points, there is no $\mathbb{R}$-action on moduli spaces of such objects.*

Define a map $f : CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)$

\[ f(x \otimes y) = \sum_{w \in L_0 \cap L_2} \sum_{\psi \in \Pi_2(x, y, w)} \# M(\psi) \omega \]

where $M(\psi) = 0$

Lemmm: $f$ is a chain map.
Pf: Let $\psi \in \pi_2(x, y, z)$, $\mu(\psi) = 1$.

Then

$$\partial M(\psi) = \left( \text{counted by coeff. of } z \text{ in } \partial f(x \otimes y) \right) \sqcup \left( \text{counted by coeff. of } z \text{ in } f(x \otimes y) \right) \sqcup \left( \text{counted by coeff. of } z \text{ in } f(x \otimes y) \right)$$

So, $\partial f + f \partial = 0$.

(Recall: $M(\psi)$ is a cpt oriented 1-mfd.)

ex: $(\Sigma, \alpha, \beta, \gamma), (\Sigma, \alpha, \delta, \zeta)$, $\beta_i \mapsto \delta_i$. Hamiltonian isotopy:

Then $(\Sigma, \beta, \delta, \zeta)$ is weakly admissible and represents $\gamma_g := (S^2 \times S^1) \# \ldots \# (S^2 \times S^1)$. What is $\hat{HF}(\gamma_g)$?

e.g. $g = 2$:

So, $\partial = 0$, and

$\hat{HF}(S^2 \times S^1 \# S^2 \times S^1) 
\cong H_*(S^1) \otimes H_*(S^1)$.
For general $g$, we'll have that
\[
\widehat{HF}(Y_g) \cong (H_*(S'))^g
\]
where the maximal degree element is represented by the cycle \( \Theta_{g^g} = \{\Theta_1, \ldots, \Theta_g\} \) from previous diagram.

Rmk: Even more generally,

**Thm:** \( \widehat{HF}(\mathcal{H}_1 \# \mathcal{H}_2) \cong \widehat{HF}(\mathcal{H}_1) \otimes \widehat{HF}(\mathcal{H}_2) \)

Ex: Let \( \beta \rightarrow \gamma \) be a handleslide, i.e. let \( \gamma_i \) differ from \( \beta_i \) by Ham. isotopy if \( i \neq 1 \) and let \( \gamma_i \) be the result of a handleslide of \( \beta_i \) over \( \beta_2 \):

\[
\begin{array}{c}
\circ \quad \Theta_1 \quad \gamma_1 \\
\oplus \quad \Theta_2 \quad \gamma_2
\end{array}
\]

\( (\Sigma, \beta, \gamma, z) \) again represents \( S^2 \times S^1 \)

Claim: (will prove later): \( \Theta_{g^g} \) is a cycle representing the top-degree generator of \( \widehat{HF}(\#_g S^2 \times S^1) \cong (H_*(S'))^g \)

(Shown here for \( g = 2 \))
Associativity of triangles:

There's an analogous notion of Whitney quadrilaterals, and we collect their homotopy classes into sets
\[ \pi_2(x, y, w, v) \]

* If \( \eta \in \pi_2(x, y, w, v) \) and \( M(\eta) \neq \emptyset \), then
\[ \dim(M(\eta)) = m(\eta) + 1 \]

We then define a map
\[ h_{\alpha \beta \gamma \delta} : \overset{\wedge}{\text{CF}}(H_{\alpha \beta}) \otimes \overset{\wedge}{\text{CF}}(H_{\beta \gamma}) \otimes \overset{\wedge}{\text{CF}}(H_{\gamma \delta}) \rightarrow \overset{\wedge}{\text{CF}}(H_{\alpha \delta}) \]

\[ h_{\alpha \beta \gamma \delta}(x \circ y \circ w) := \sum \sum \sum (\# M(\phi)) n \]

\[ \forall \pi_\alpha \cap \pi_\gamma \phi \in \pi_2(x, y, w, v) \]
\[ m(\phi) = -1 \]
\[ n_\pi(\phi) = 0 \]

Recall we also had \( f : \overset{\wedge}{\text{CF}}(H_{\alpha \beta}) \otimes \overset{\wedge}{\text{CF}}(H_{\beta \gamma}) \rightarrow \overset{\wedge}{\text{CF}}(H_{\alpha \gamma}) \)

\[ f_{\alpha \beta \gamma}(x \circ y) := \sum \sum \sum (\# M(\phi)) n \]

\[ \forall \pi_\alpha \cap \pi_\gamma \phi \in \pi_2(x, y, w) \]
\[ M(\phi) = 0 \]
\[ n_\pi(\phi) = 0 \]

(Also have \( f_{\alpha \beta \delta} \), etc.)

Thm: \( h_{\alpha \beta \gamma \delta} \) is a chain htpy between the maps \( f_{\alpha \beta \delta} \cdot f_{\beta \gamma \delta} \cdot f_{\alpha \delta} \) and \( f_{\alpha \beta \gamma} \cdot f_{\beta \gamma \delta} \cdot f_{\alpha \delta} \).

68
**Pf:** Let \( \phi \in \pi_2(x, y, v, w) \), \( \mu(\phi) = 0 \).

\[
\delta M(\phi) = \begin{pmatrix}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{pmatrix} \quad \begin{pmatrix}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{pmatrix}
\]

* The 1\textsuperscript{st}, 3\textsuperscript{rd}, 4\textsuperscript{th} types of configurations are those counted by the coefficient of \( w \) in \( h_{\alpha \beta \gamma \delta} (\phi(x \otimes y \otimes v)) \).

* The 2\textsuperscript{nd} type are counted by the coefficient of \( w \) in \( \delta (h_{\alpha \beta \gamma \delta} (x \otimes y \otimes v)) \).

* The 5\textsuperscript{th} type are counted by the coefficient of \( w \) in \( f_{\chi \delta \gamma \delta} (f_{\alpha \beta \delta} (x \otimes y) \otimes v) \).

* The 6\textsuperscript{th} type are counted by the coefficient of \( w \) in \( f_{\alpha \beta \delta} (x \otimes f_{\beta \delta} (y \otimes v)) \).

Since \( \delta M(\phi) \) is cpt. of dimension 0, the result follows.

(we have suppressed signs coming from orientations of moduli spaces as usual).
Now define a map \( F_{\alpha \beta} : \hat{HF}(H_{\alpha \beta}) \rightarrow \hat{HF}(H_{\alpha \gamma}) \) by
\[
F_{\alpha \beta}(x) = (f_{\alpha \beta})^*(x \otimes \Theta_{\beta \gamma})
\]
(and similarly define \( F_{\alpha \gamma}, F_{\alpha \delta}, \) etc.)

**Lemma 1:** \( f_{\beta \gamma \delta}(\Theta_{\beta \gamma} \otimes \Theta_{\gamma \delta}) = \Theta_{\beta \delta} \) \( \Rightarrow F_{\beta \gamma \delta}(\Theta_{\beta \delta}) = \Theta_{\beta \delta} \)

**Pf:** (as indicated)

Zooming in on the dotted region (or one of its twins on another handle),

(Labels regions as indicated)

We want to find possible domains of classes
\( \phi \in \pi_2(\Theta_{\beta \gamma}, \Theta_{\gamma \delta}, u) \)
where
\[
\begin{align*}
\mu(\phi) &= 0, \\
n_2(\phi) &= 0, \\
\Delta(\phi) &= 0
\end{align*}
\]
Note that in the previous picture,

- $R^i_7 = R^i_0 = R^2_0$ for $i = 1$ and $i \geq 3$, and
- $R^2_7 = R_5^2$

Let $D_i$ have coefficient $n^i_j$ in $R^i_j$ (and assume that $n^i_0 = 0$, $n^i_j \geq 0$ for all $i, j$).

Notice that boundary conditions stipulate:

**(β):** $\partial (\partial D_i) = u^i - \Theta^i_{βr}$

**(γ):** $\partial (\partial D_i - \Theta^i) = \Theta^i_{βr} - \Theta^i_{γs}$

**(δ):** $\partial (\partial D_i - \Theta^i_{δs}) = \Theta^i_{γs} - u^i$

By (δ), we see that for $i \geq 3$:

$$n^i_3 = n^i_4 - n^i_2 = n^i_7 - n^i_5 + 1 = n^i_1 - n^i_6 + 1 \quad (\text{for each } i)$$

We obtain more equations by imposing (β), (δ) in two cases (for each $i$). First fix $i \geq 3$.

**Case 1:** $u^i = n^i_{β8}$. Then:

**(β)** $n^i_4 = n^i_3 - n^i_2 + 1 = n^i_7 - n^i_5 + 1$

**(δ)** $n^i_2 = n^i_3 - n^i_1 = -n^i_6 + 1 = n^i_3 - n^i_4 + 1$

After some work, we obtain:

- Either $n^i_2 = n^i_3 = n^i_4 = 1$, $n^i_1 = n^i_5 = n^i_6 = 0$
- Or $n^i_4 = n^i_5 = n^i_6 = 1$ and $n^i_1 = n^i_2 = n^i_3 = 0$
Case 2: \( u^i = \Theta^i \). Then:

\[
\begin{align*}
(p) & \implies n^i_y = n^i_3 - n^i_2 = n^i_7 + 1 = n^i_6 - n^i_5 + 1 \\
(y) & \implies n^i_3 - n^i_1 = n^i_6 + 1 = n^i_3 - n^i_4 + 1 = n^i_2 + 1
\end{align*}
\]

After some work, we see

\[
\begin{align*}
n^i_3 = n^i_4 = 1 \quad &\text{and} \quad n^i_1 = n^i_2 = n^i_5 = n^i_6 = 0
\end{align*}
\]

Now we study \( i \in \{1, 2\} \):

Case 1: \( \Theta\Theta^2 \):

We have:

\[
\begin{align*}
n^i_4 = n^i_3 - n^i_2 + 1 &= n^i_7 + 1 = n^i_6 - n^i_5 + 1 \\
n^i_2 = n^i_3 - n^i_1 &= n^i_6 + 1 = n^i_3 - n^i_4 + 1 \\
n^i_2 - n^i_5 = n^i_3 - n^i_2 + 1 &= n^i_2 - n^i_5 + 1 = n^i_6 - n^i_5 + 1 \\
n^i_2 - n^i_6 = n^i_3 - n^i_1 &= n^i_6 + 1 = n^i_3 - n^i_4 + 1
\end{align*}
\]

(\(*\)) \implies either

\( a \): \( n^i_2 = n^i_3 = n^i_4 = 1, \quad n^i_1 = n^i_5 = n^i_6 = 0 \)

\( b \): \( n^i_1 = n^i_2 = n^i_3 = n^i_5 = n^i_6 = 1, \quad n^i_1 = n^i_2 = n^i_3 = 0 \)

In conjunction with (\(*\)), we see that only \( a \) is possible, and either \( n^2_2 = n^2_3 = n^2_4 = 1 \) other \( n^2_1 = n^2_5 = n^2_6 = 0 \)

or \( n^2_1 = n^2_5 = n^2_6 = 1, \) all other \( n^2_i = 0 \).

i.e. we have \( +1 + 1 \) or \( +1 + 1 + +1 \)

Case 2: \( \Theta n^2 \quad \rightarrow \quad +1 + +1 \)

Case 3: \( n^i \Theta^2 \quad \rightarrow \quad +1 + +1 \)

Case 4: \( n^i n^2 \quad \rightarrow \quad +1 + +1 \)

\( \{ \text{exercise}\} \)
Now notice that there is an index-0 class $\phi_0 \in \Pi_2(\Theta_{\beta \rho}, \Theta_{\gamma \sigma}, \Theta_{\beta \delta})$, corresponding to having \( 1 \leq \text{index} \) in every component. (and in this is the only index-0 class in any) $\Pi_2(\Theta_{\beta \rho}, \Theta_{\gamma \sigma}, u)$

Indeed,

(i) $\bigcirc = \bigcirc + \bigcirc$ and

(ii) $\bigcirc = \bigcirc + \bigcirc$

So, if $\phi \in \Pi_2(\Theta_{\beta \rho}, \Theta_{\gamma \sigma}, u)$ is not equal to $\phi_0$, its index is at least 1.

Now we study $M(\phi_0)$:

1. By Riemann Mapping theorem, there is a unique holomorphic map $U_S : S \to \Sigma$ with "image" $D(\phi_0)$ (counted with multiplicity). This is the obvious one:

$$S = \bigcirc \bigcirc \bigcirc \cdots \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \rightarrow \bigcirc + \bigcirc + \bigcirc + \bigcirc + \bigcirc + \bigcirc + \bigcirc + \bigcirc$$

2. There is only one holomorphic $g$-fold branched cover $U_D : S \to D$ — the unbranched one.

So, $M(\phi_0) = 3$ points, and we can orient it so that

$$f_{\beta \rho \gamma \delta}(\Theta_{\beta \rho} \otimes \Theta_{\sigma \delta}) = \Theta_{\rho \delta}$$
Lemma 2: \( F_{\alpha \beta \gamma} \circ F_{\alpha \beta \delta} = F_{\alpha \beta \delta} \) (similarly \( F_{\alpha \beta \gamma} \circ F_{\alpha \delta \beta} = F_{\alpha \delta \beta} \))

Pf: Associativity + Lemma 1. \( \square \)

Lemma 3: \( F_{\alpha \beta \delta} \) is an isomorphism \((F_{\alpha \beta \delta})\)

Idea of pf: The map \( F_{\alpha \beta \delta} \) is induced by \( f = f_{\alpha \beta \delta} \circ \Theta_{\beta \delta} \). There's a filtration called the "area filtration" on \( \hat{CF} \) — the important observation is that \( f = f_0 + \lambda \), where

- \( f_0 \) is filtration-preserving and \( f_0(x) = x' \) (clearly an isomorphism of chain groups)
- \( \lambda \) strictly lowers filtration

General nonsense fact: Let \( F: A \to B \) be a map of filtered groups such that \( F = F_0 + \lambda \), where \( F_0 \) is a filtration-preserving isom. and \( \lambda(x) \leq F_0(x), \forall x \in A \). Then if the filtration on \( B \) is bounded below, \( F \) is an isomorphism of groups.

So, \( F_{\alpha \beta \gamma} \) is an isomorphism! (we're done)
Return to example 2 calculation:

For notational convenience, assume $g = 2$ and work in $\text{Sym}^2(\Sigma_2)$:

There are 5 domains $D$ with $m(D) = 1$ and $D \geq 0$:

- each carry unique holomorphic reps. (modulo R-shift)
- how about these?

We'll find that exactly one of the annuli above has $M \neq \emptyset$ (which one depends on $J$).

We haven't discussed orientation on $M$, but signs work out and $\partial(\partial\Theta_2) = 0$.

(\text{in fact, the complex is}

\[
\begin{pmatrix}
\Theta_1 \Theta_2 \\
\Theta_1 n_2 \\
n_1 \Theta_2 \\
n_1 n_2
\end{pmatrix}, \text{i.e. } \partial = 0
\)
Recall: There is a correspondence:

\[ (D^2 \rightarrow \text{Sym}^2(\mathbb{C})) \leftrightarrow (\mathbb{S} \xrightarrow{u_\Sigma} \Sigma) \]

Now, every open annulus in \( \mathbb{C} \) is conformally equivalent to some “standard annulus”:

\[ A_r := \{ z \in \mathbb{C} \mid 1 < |z| < r \} \quad (\text{for some } r > 1) \]

How do we encode boundary conditions?

\[ =: A(r, \theta, \rho_{\text{in}}, \rho_{\text{out}}) \]

where \( r, \theta \) are as shown,

\[ \rho_{\text{in}} = \text{inner conformal ratio} = \frac{\text{length of } g_{\text{in}}}{\text{length of } b_{\text{in}}} \]

\[ \rho_{\text{out}} = \text{outer conformal ratio} = \frac{\text{length of } g_{\text{out}}}{\text{length of } b_{\text{out}}} \]

So, \( \dim \left( \mathcal{M} \left( \Sigma_4-\text{punctured standard annuli}^3 \right) / \text{rotation} \right) = 4 \)

Lemma: \( A(r, \theta, \rho_1, \rho_2) \xrightarrow{\text{hol.}} D^2 \iff \rho_1 = \rho_2 \)

So, we have:

\[ M^3_{\text{branched covers}} \supset M^1_{\text{annuli, bihol. to } D} \]

1-dimensional via slit length:

\[ M\left( \Sigma_4-\text{annuli}^3 / \text{rot} \right) \]
Look at extremal situations for slit:

- \( \text{slit} = 0 \Rightarrow \frac{P_{out}}{P_{in}} = 0 \)
- \( \text{slit reaches } \gamma \text{ again} \Rightarrow \frac{P_{out}}{P_{in}} = L \in \mathbb{R} \)
  
  \[
  \begin{cases}
  1 & \text{if } L > 1 \\
  0 & \text{if } L \leq 1
  \end{cases}
  \]

So, schematically we have:

Now back to invariance:

Let \( D(\phi) = D_1 + D_2 \)
\( D(\psi) = D_1 + D_3 \)

Then \( \# M(\phi), \# M(\psi) \in \{0, 1\} \)

**Lemma:** \( \# M(\phi) + \# M(\psi) = 1 \)

**Idea:**

So, exactly one of \( \phi, \psi \) has holomorphic representative, and so \( \partial = 0 \).
Combinatorial Description of $\hat{HF}$: (Sarkar–Wang)

Lemma: Let $\phi \in \pi_2(x,y)$ have $m(\phi) = 1$, and let $D(\phi) = \sum_i a_i R_i$ where $D(\phi) \geq 0$ and $a_i > 0 \Rightarrow R_i$ is a bigon ($\bigcirc$) or a rectangle ($\square$).

Then $M(\phi)$ has one point.

Proof: $m(\phi) = e(D) + n_x(D) + n_y(D)$

$e(\bigcirc) = \frac{1}{2}$ and $e(\square) = 0$ ($n_x + n_y > 0$)

Two cases:

1. $e(D) = \frac{1}{2} \Rightarrow n_x(D) + n_y(D) = \frac{1}{2}$

   $\Rightarrow D = \bigcirc$, one representative

2. $e(D) = 0$, $n_x(D) + n_y = 1$

   $\Rightarrow D = \square$, one representative

Remark: If $g(\Sigma) > 2$, there can't exist a Heegaard diagram with all regions bigons or squares.

(Note: $2 - 2g = \chi(\Sigma) = e(\Sigma) = \sum e(R_i)$)

To remedy this, (since domains counted by $\hat{HF}$ avoid $z$), we allow the $z$-region to be more complicated.
Thm: Let $H = (\Sigma, \alpha, \beta, z)$. We can modify $H$ via a sequence of isotopies and handleslides avoiding $z$ to reach $H'$, a diagram in which all non-$z$ regions are bigons or rectangles.

Pf. Step 1: Via isotopies, modify diagram so that all regions are polygons, e.g.

\[
\begin{array}{c}
\alpha \\
\beta \\
\alpha \\
\beta \\
\alpha \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
\alpha' \\
\beta' \\
\alpha' \\
\beta' \\
\alpha' \\
\end{array}
\]

Step 2: A region $R$ is a polygon with $2n$ edges. If $n \leq 2$, we say $R$ is good; otherwise, $R$ is bad.

The badness of $R$ is $b(R) = \max \{0, n-2\}$

($R$ good $\iff$ $b(R) = 0$)

Goal: move all badness to the region containing $z$. Let $b(R) > 0$. Then let $d(R)$ be the minimum number of times we must cross a $\beta_i$ to reach the $z$-region. ("distance from $z" \)
Let $d(N)$ be the maximum distance between $x$ and a bad region.

Let $D_1, \ldots, D_m$ be the d-bad regions ordered as

\[ b(D_1) \geq b(D_2) \geq \ldots \geq b(D_m) \]

\[ \forall d(N) := \sum_{i=1}^{m} b(D_i) \]

The $d$-complexity is a tuple:

\[ C_d(N) = \left( \sum_{i=1}^{m} b(D_i), -b(D_1), -b(D_2), \ldots, -b(D_m) \right) \]

(order possible such tuples lexicographically)

Here's how to execute the algorithm:

Choose $D_*$, $\beta$-adjacent to $D_m$ and w/ distance $d-1$.

Try the following finger move:

(i.e. always push $b_*$ into)

(Dm and out across $a_2$

Continue pushing the finger subject to these rules:

1. If a bijon is reached, stop.

2. If finger enters a rectangle, keep pushing across unless its of distance $< d$ (else stop):

\[ \begin{array}{c}
\text{\bar{}}
\end{array} \Rightarrow \begin{array}{c}
\text{\bar{}}
\end{array} \]
3. It finger enters another bad region, stop.

4. If finger returns to $D_m$, be more careful:
   a. If it returns via $a_1$ or $a_3$, abandon the finger move and instead do the obvious $h$-slide:
      ![Diagram]
   b. If it returns to $D_m$ via $a_k$ ($3 < k \leq n$), we abandon finger move and try another of $b_+$ across $a_3$.

Lemma: Continuing this process, we'll either find a non-returning finger or a finger leaving from $a_1$ and returning via $a_j$, $j > k$.

(In former case, use this finger. In latter, use $h$-slide as in (a) above)

Call the result after an iteration $\tilde{H}$.

Lemma: (exercise) One of the following holds

(i) $d(\tilde{H}') < d(\tilde{H})$
(ii) $d(\tilde{H}') = d(\tilde{H})$ but $b_d(\tilde{H}') < b_d(\tilde{H})$
(iii) $d(\tilde{H}') = d(\tilde{H})$ and $b_d(\tilde{H}') = b_d(\tilde{H})$, but total # of "bad" regions decreases.

So, eventually, we'll obtain a diagram $\tilde{H}$ with $d(\tilde{H}) = 0$. 
Knots: Let \( K \subset S^3 \) be a knot or link (with orientation).

**Def.** The **Alexander-Conway polynomial** of \( K \) is the unique polynomial \( \Delta_K \in \mathbb{Z} \left[ q^{\frac{1}{2}}, q^{-\frac{1}{2}} \right] \) satisfying

1. \( \Delta(0) = 1 \)
2. \( \Delta \left( q^\sigma \right) - \Delta \left( q^{-\sigma} \right) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \Delta \left( q^\sigma \right) \)

It's an invariant of oriented links.

**ex:** 
\[
\Delta(00)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = \Delta(00) - \Delta(00) = 1 - 1 = 0
\]

\[\Rightarrow \Delta(00) = 0\]

(in general, \( \Delta(L) = 0 \) if \( L \) is split link)

2. \( \Delta \left( q^\sigma \right) - \Delta \left( q^{-\sigma} \right) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \Delta \left( q^\sigma \right) \)

\[\Rightarrow \Delta \left( q^\sigma \right) = q^{\frac{1}{2}} - q^{-\frac{1}{2}}\]

3. \( \Delta \left( q^\sigma \right) - \Delta \left( q^{-\sigma} \right) = \Delta \left( q^\sigma \right) \left( q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right) \)

\[\Rightarrow \Delta \left( q^\sigma \right) = q - 1 + q^{-1}\]

**Properties:**
1. \( \Delta_K \in \mathbb{Z} \left[ q, q^{-1} \right] \)
2. \( \Delta_K(1) = 1 \)
3. \( \Delta_K(q) = \Delta_K(q^{-1}) \)

We want to "categorify \( \Delta_K \), i.e. produce a bigraded group \( \widehat{HFK}^i_\ast(K,s) \) with

\[
\Delta_K(q) = \sum_{i,s \in \mathbb{Z}} (-1)^i q^s \text{rk} \left( \widehat{HFK}^i_\ast(K,s) \right)
\]
Recall: $\hat{HF}(S^3) \cong \mathbb{Z}$

Choose some diagram $H = (\Sigma; \alpha; \beta; w)$ for $S^3$.

Since $\text{Spin}^c(S^3) = H_1(S^3) = 0$, we can find $\phi \in \pi_2(x,y)$

for any pair $x, y \in \pi_1 \cap \pi_1 \beta$. There is thus a

relative $\mathbb{Z}$-grading on $\hat{CF}(H)$:

$$\text{gr}(x,y) = \mu(\phi) - 2n_\omega(\phi) \in \mathbb{Z}$$

Diagrams for Knots:

Fix 2-pointed diagram $(\Sigma_g; \alpha; \beta; w, z)$, where

$(\Sigma; \alpha; \beta)$ is a diagram for $S^3$ and $w, z \in \Sigma \setminus \alpha \setminus \beta$.

Such a diagram induces a Knot $K \subset S^3$:

We say that $(\Sigma, \alpha, \beta, w, z)$ is a doubly-pointed Heegaard diagram for $K \subset S^3$.

(we can see $K$ on the surface $\Sigma = \text{connect } w \text{ to } z \text{ via an arc in } \Sigma \setminus \alpha \text{ and then } z \text{ back to } w \text{ via an arc in } \Sigma \setminus \beta$)

Lemma: Such a diagram exists for any knot $K \subset S^3$.

Pf: Let $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_{g-1})$ represent $S^3 \setminus N(K)$.

Now let $\beta_g$ be a curve on $\Sigma_g$ representing the meridian of $K$. 

83
Then place \( z, w \) on either side of \( \beta_g \), away from the \( \sigma \) curves.

**Proposition**: If \( \mathcal{H} \) and \( \mathcal{H}' \) represent \( K \subset S^3 \), then they are related by a sequence of isotopies, stabilizations, and handleslides (where isotopies, handleslides avoid \( w, z \))

**Def.** \( \hat{HF}(S^3, K) = HF(T_\alpha, T_\beta) \) in \( \text{Sym}^3(\Sigma - w - z) \)
(i.e. \( \delta \) only counts \( \phi \) with \( n_\Sigma(\phi) = n_W(\phi) = 0 \))

"Forgetting" \( w \) or \( z \) gives \( \hat{HF}(S^3) \)

**Decomposition**: We say \( x \sim y \) iff \( \exists \phi \in \pi_2(S^3) \) with \( n_\Sigma(\phi) = n_W(\phi) = 0 \).
Then the set of equivalence classes is

\[ \hat{H}_1(S^3 - N(K)) = \mathbb{Z} \] (generated by meridian of \( K \))

\[ \hat{HF}(K) = \bigoplus_{s \in \mathbb{Z}} \hat{HF}(K, s) = \bigoplus_{i, s \in \mathbb{Z}} \hat{HF}_{s, i}(K, s) \]

Here \( s \) is the **Alexander grading** (\( \text{Spin}^c \)-structures) and \( i \) is the **Maslov grading**

(we fix Maslov grading by declaring that \( \hat{HF}(S^3) \) is supported in \( i = 0 \))
Properties of $\hat{HF}_K$:

1. $\Delta_K(q) = \bigoplus_{i,s \in \mathbb{Z}} (-1)^i q^s \text{rk}(\hat{HF}^i; K, s)$

2. Classical result: $g(K) = \max \{ s \mid a_s \neq 0 \}$
   
   (\( \Delta_K(q) = \sum_{s \in \mathbb{Z}} a_s q^s \))

   But this isn't always equality!

   Thm: (Ozsváth-Szabó) $g(K) = \max \{ s \mid \hat{HF}^i(K, s) \neq 0 \}$

   Thm: (Ni) $S^3 \setminus N(K)$ fibers over $S^3$ ("$K$ is fibered")
   
   if and only if $\hat{HF}_K(K, g(K)) \cong \mathbb{Z}$.

3. $\hat{HF}^i(K, s) = \hat{HF}^{i-2s}(K, -s)$

   $\hat{HF}^i(K, s) = \hat{HF}^{i}(K^r, s)$

   $\hat{HF}^i(K, s) = \hat{HF}^{-i}(\overline{K}, -s)$

We can use a bridge presentation of $K$ to produce a Heegaard diagram for $K$

( and subsequently compute $\hat{HF}_K(K)$ )

ex:

$T = $ trefol

$\hat{CFK}(T) = \mathbb{Z}\langle a, b, c \rangle \cong \mathbb{Z}^3$, $d = 0$ (all disks hit $w$ or $z$)

$\hat{CFK}(T) = \mathbb{Z}^3$

So, $\hat{HF}_K(T) = \mathbb{Z}^3$
Gradings: Let \( \phi \in \pi_2(x, y) \)

(i) Maslov grading \( M \)
\[
M(x) - M(y) = M(\phi) - 2n_w(\phi)
\]
(this is the homological grading on \( \hat{CF}(\mathcal{U}) \))
when one forgets the basepoint \( \mathcal{Z} \)

(ii) Alexander grading \( A \)
\[
A(x) - A(y) = n_x(\phi) - n_w(\phi)
\]

Note: (i) \( \mathcal{Z} \) preserves \( A \), decreases \( M \) by 1
(ii) We can fix \( M, A \) to be absolute
gradings by requiring that \( \hat{HF}(S^3) \) (forget \( \mathcal{Z} \)) is
supported in \( M = 0 \) and requiring that
\[
\chi(\hat{HF}) = \Delta_k \text{ is symmetric. }
\]

\( \text{ex: } \hat{CFK}(T) = \mathbb{Z} \langle a, b, c \rangle. \)
There is \( \phi \in \pi_2(b, c) \) of index 1:
So, \( M(b) - M(c) = 1 - 0 = 1 \)
\( A(b) - A(c) = 1 - 0 = 1 \)
There's \( \psi \in \pi_2(b, a) \) of index 1:
\( M(b) - M(a) = 1 - 2 = -1 \)
\( A(b) - A(a) = 0 - 1 = -1 \)
Now forgetting \( \mathcal{Z} \), we see that \( \hat{HF}(S^3) \cong \langle a \rangle \)
So, we have
\[
\hat{HFK}(T) = \begin{align*}
\begin{array}{c|cc}
\phi & M & A \\
\hline
a & 0 & k \\
b & -1 & k-1 \\
c & -2 & k-2 \\
\end{array}
\end{align*}
\]
\[ \times \Delta_T = \Delta_T^-, \text{ but } \hat{HFK}(T) \neq \hat{HFK}(\mathcal{T}) \text{ (gradings are different) } \]
\( \text{ex: } F = \text{Figure-eight knot} \)

Again, no disks avoid both \( z \) and \( w \). So, \( \mathcal{D} = 0 \).

\[ \hat{\hbox{HFK}}(F) \cong \mathbb{Z}^5 \]

\[
\begin{align*}
M(c) - M(d) &= 1 - 2 = -1, \quad A(c) - A(d) = -1 \\
M(b) - M(a) &= 1 - 2 = -1, \quad A(b) - A(a) = -1 \\
M(c) - M(b) &= 1, \quad A(c) - A(b) = 1 - 0 = 1 \\
M(b) - M(e) &= -1, \quad A(b) - A(e) = -1
\end{align*}
\]

\( \hat{\hbox{CF}}(\mathcal{H}_w) \) has the form \( \langle a \rangle \cap \langle c \rangle \), so \( \hat{\hbox{HFK}}(\mathcal{H}_w) = \langle a \rangle \cap \langle c \rangle \cap \langle d \rangle \cap \langle e \rangle \cap \langle b \rangle \cap \langle \mathcal{H}_w \rangle \).

So, we have

<table>
<thead>
<tr>
<th></th>
<th>( M )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>( k )</td>
</tr>
<tr>
<td>b</td>
<td>-1</td>
<td>( k-1 )</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>( k )</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>( k+1 )</td>
</tr>
<tr>
<td>e</td>
<td>0</td>
<td>( k )</td>
</tr>
</tbody>
</table>

\( \Rightarrow k = 0 \)

\[ \hat{\hbox{HFK}}(F) : \]

\[ A \] and \[ M \] are connected.

Note: \( F = \overline{F} \), and indeed \( \hat{\hbox{HFK}}(F) = \hat{\hbox{HFK}}(\overline{F}) \)

Remark: We see \( g(T) = g(F) = 1 \), further, \( A - M \) is constant!
Thm: (Rasmussen) When \( K \) is 2-bridge, 
\( A-M \) is constant on \( \widehat{HFK} \) and in fact 
\[ A-M = \frac{\sigma(K)}{2} \] (\( \sigma \) is classical knot signature)

In other words, 
\[ \widehat{HFK}_4(K,s) = \begin{cases} 
\mathbb{Z} \left[ a_s \right] & \text{if } t=s+\frac{q}{2} \\
0 & \text{else} 
\end{cases} \]

Such knots are called \( \text{Floer } \sigma \)-thin.

Thm (Oszváth-Szabó): Alternating knots are thin.

Heegaard diagrams with many basepoints:
We can "trade" genus for basepoints:
A multi-pointed Heegaard diagram is a tuple:
\[(\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_g+k^2\}, \beta = \{\beta_1, \ldots, \beta_g+k^2\}, z_1, \ldots, z_{k+1})\]

where
- \( \alpha \) span \( g \)-dim'l subspace of \( H_1(\Sigma) \)
- \( \beta \)
- Each component of \( \Sigma \setminus (\bigcup \alpha_i) \) has exactly one \( z_j \)
- \( \Sigma \setminus (\bigcup \beta_i) \)

Such a diagram determines a Heegaard splitting of some \( Y; \)
\[ U_\alpha = (\sum \times [0,1]) \cup (2g+K \text{handles}) \cup (k+1 \text{ 3-balls}) \]
\[ \text{attached along } \alpha_i \times \mathbb{S}^3 \] (the \( \beta \) is similar)
One can then compute \( \text{Log. Floer homology} \) of \( \Pi_\alpha = \alpha_1 \times \ldots \times \alpha_{g+k}, \quad \Pi_\beta = \beta_1 \times \ldots \times \beta_{g+k} \) in \( \text{Sym}^{g+k}(\Sigma - \{ z_1, \ldots, z_{k+1}, z \}) \subset \text{Sym}^{g+k}(\Sigma) \).

**Proposition:** \( \text{HF}(\Pi_\alpha \times \Pi_\beta) \cong \widehat{\text{HF}}(Y) \otimes (\widehat{H^*(S')} \otimes \mathbb{C}) \) 

"\( 2^k \)-copies of \( \widehat{\text{HF}}(Y) \)"

**Pf:** Fact: any two multi-pointed Heegaard diagrams for \( Y \) are connected by a sequence of usual moves along with a new kind of "(de)-stabilization":

\[
\xymatrix{ \mathbb{Z}_k \ar[r] & \mathbb{Z}_{k+1} \\
\mathbb{Z}_{k+1} \ar@{.>}[u] & \mathbb{Z}_k \ar@{.>}[u]
}
\]

On generators, \( \{ x_1, \ldots, x_{g+k} \} \mapsto \{ x_1, \ldots, x_{g+k}, y_1 \} \ugging \{ x_1, \ldots, x_{g+k}, y_2 \} \)

Also, coefficient of \( \{ x_1, \ldots, x_{g+k}, y_2 \} \) in \( \partial \{ x_1, \ldots, x_{g+k}, y_1 \} \) is zero (because of two cancelling disks, e.g.)
Let \( L \subset S^3 \) have \( l \) components. A Heegaard diagram for \( L \) is
\[
(\Sigma, \alpha_1, \ldots, \alpha_{g+l-1}, \beta_1, \ldots, \beta_{g+l-1}, w_1, \ldots, w_{\ell}),
\]
where \((\Sigma, \alpha, \beta, z)\) and \((\Sigma, \alpha, \beta, w)\) are multi-pointed diagrams for \( S^3 \).

If \( A_i \) (resp. \( B_i \)) are connected components of \( \Sigma \setminus (\bigcup_j x_j) \) (resp. \( \Sigma \setminus (\bigcup_i \beta_i) \)), then \( z_k, w_k \in A_k \cap B_k \).

Connecting (for each \( i \)) \( w_i \) to \( z_i \) in \( U_\alpha \)
and \( z_i \) to \( w_i \) in \( U_\beta \) via unknotted arcs
yields \( L \subset S^3 \).

**Def:** \( \hat{\text{HF}}(L) = \text{HF}(\Pi_\alpha, \Pi_\beta) \circ_{\text{Sym}} (\Sigma \setminus \bigcup \{w_i, z_i, x_j, \beta_i \}) \)

\[
\hat{\text{HF}}(L) = \bigoplus_{i,s \in \mathbb{Z}} \hat{\text{HF}}_{i,s}(L, \delta), \quad \delta \text{ is Maslov grading } "M" \text{ and } s \text{ is Alexander grading } "A" \text{ (defined as in knot case)}
\]

**Thm:** \( \chi(\hat{\text{HF}}(L)) = (1-q_i^{-1})^{l-1} \Delta_L(q) \)

Alexander polynomial for \( L \).
e.g.

1. $L = \infty$ ($\Delta_L = 0$)
   can use $g = 0$ ($g + l - 1 = 1$)

\[
\hat{CFK}(L) = \mathbb{Z}^{2(a,b)}, \quad d = 0 \Rightarrow \hat{HFK}(L) = \mathbb{Z}^2
\]

$M(b) - M(a) = 1 - 2 = -1, \quad A(a) - A(b) = 0$

2. $L = \infty$
   \[
   \left( \Delta_L = g^{1/2} - g^{-1/2}, \quad \text{so} \right)
   \left( 1 - g^{-1} \right) \Delta_L = g^{1/2} - 2g^{-1/2} + g^{-3/2}
   \]

$g = 0$ ($\Rightarrow g + l - 1 = 1$).

\[
\hat{HFK}(L) = \mathbb{Z}^4 \quad (\text{exercise: compute } M, A)
\]
$\hat{HF}$ with several basepoints:
$k \geq l = \# \text{ of components of } L \subset S^3$

with Heegaard diagram $(\Sigma_g; \alpha_1, ..., \alpha_{g+k-1}, \beta_1, ..., \beta_{g+k-1}, w_1, ..., w_k)

$HF(T_\alpha, T_\beta) \cong \hat{HF}_K(L) \otimes \left( H_\ast(S^1) \otimes (k-l) \right)$

$\operatorname{Sym}^k \left( \mathbb{Z} \langle w_1, ..., w_k, z_1, ..., z_k \rangle \right)$

Knots: $L = 1$

$(\Sigma_g, \alpha_1, ..., \alpha_{g+k-1}, \beta_1, ..., \beta_{g+k-1}, z_1, ..., z_k; w_1, ..., w_k)$

$K = \text{union of } 2k \text{ intervals connect } w_i \text{ to } z_i \text{ in } U_\alpha,$

$z_i \text{ to } w_{i+1} \text{ in } U_\beta$

$(w_{k+1} := w_1)$

$HF(T_\alpha, T_\beta) = \hat{HF}_K(K) \otimes \left( V \otimes (k-l) \right)$

$V := H_\ast(S^1) = \mathbb{Z}_{(-1, -1)} \oplus \mathbb{Z}_{(0, 0)}$

Any two multipointed diagrams for $K$ can be connected by sequence of usual moves and new stabilization:

 $$ \begin{pmatrix} w_i \end{pmatrix}^{K} \quad \longleftrightarrow \quad \begin{pmatrix} w_1 \end{pmatrix}^{(k-1)}$$

$$U_\alpha \quad \longleftrightarrow \quad U_\beta$$

$w_1, w_2$
We use the same bigrading $M, A$

(where $n_w(\phi) = \sum_i n_{w_i}(\phi)$, $n_z(\phi)$ similar)

Advantage: Any knot admits a multi-pointed Heegaard diagram of genus 0.

ex:

\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) circle (1);
\draw[thick] (0,0) circle (0.5);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\draw[thick,blue] (0,0) circle (2);
\draw[thick,red] (0,0) circle (1);
\draw[thick] (0,0) circle (0.5);
\end{tikzpicture}
\end{center}

(continued on next page)

Good diagrams $B$ or $C$.

\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) circle (1);
\draw[thick] (0,0) circle (0.5);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\draw[thick,blue] (0,0) circle (2);
\draw[thick,red] (0,0) circle (1);
\draw[thick] (0,0) circle (0.5);
\end{tikzpicture}
\end{center}
Combinatorial Knot Floer Homology

Grid diagrams: Given $K \subset S^3$ (a knot), a $n \times n$ grid diagram for $K$ is an $n \times n$ grid with $2n$ markings of two types ($W$, $Z$), s.t.

1. Each row has exactly one $W$ and one $Z$
2. Each column has exactly one $W$ and one $Z$
3. Joining $W$ to $Z$ by vertical arcs and $Z$ to $W$ by horizontal (vertical always overpasses) gives a diagram for $K$.

E.g., $n = 5$

<table>
<thead>
<tr>
<th>W</th>
<th>Z</th>
<th>W</th>
<th>Z</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>W</td>
<td>Z</td>
<td>W</td>
<td>Z</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
<td>Z</td>
<td>Z</td>
<td>Z</td>
</tr>
</tbody>
</table>

Proposition: Every knot has a grid diagram

Pf: 1. First isotope all arcs to be vertical or horizontal.
2. As needed, replace $\frac{1}{-1}$ with $\frac{1}{-1}$
3. Split into grid (grid size will depend on number of corners).
Identifying opposite edges of a grid diagram for $K$ gives a genus-1 Heegaard diagram for $K$ with $n$ $\alpha$-curves, $n$ $\beta$-curves, $2n$ basepoints.

Furthermore, this diagram is nice (in Sarkar-Wang sense) so admits combinatorial calculation of $\hat{HF}_K$.

Differential only counts "empty rectangles", i.e.

Recall:

$$HF(\Pi_\alpha, \Pi_\beta) \cong \hat{HF}_K \otimes \mathbb{V}^{n-1}$$

in $\text{Sym}^n(\Sigma \setminus \{w, z\})$ ($\mathbb{V} := H^1(S')$)

$$HF = \mathbb{Z} \langle T_\alpha \cap T_\beta \rangle = \mathbb{Z} \langle \{x \mid x = \{x_1, \ldots , x_n\} \text{ for some } \sigma \in S_n \} \rangle$$

i.e. there are $n!$ generators.

$$2x = \sum \pm n_{xy} y$$, $n_{xy} =$ # of empty rectangles from $x$ to $y$

Claim: 1. $n_{xy} \neq 0 \iff x$ and $y$ differ in exactly two components

2. $K$ a knot $\Rightarrow n_{xy} \in \{0, 1, 2\}$

$\overline{a_{ij}}$
Lemma 1: \( A(\bar{x}) - A(y) = A(\overline{K}, P_{xy}) \)
\[
(p_{xy} = \exists \bar{y}, \phi \in \text{Type}(\overline{x, y}))
\]
Given \( x, y \) union of arcs connecting \( \bar{x}, \bar{y} \), \( p_{xy} \)
Computing \( A \):

Gradings:
\[
\begin{align*}
W(x) - W(y) &= W(\bar{x}) - W(\bar{y}) \\
A(x) - A(y) &= n^z(\bar{x}) - n^z(\bar{y})
\end{align*}
\]

Grid instead describes a link. If \( x, y \) are in \( K \), then the
are empty, and all of the \( z \)'s and weight \( x, y \). Hence, both
Now if my compete \( R \) and \( \overline{R} \), then something
\[
\begin{align*}
(x = y) & \quad (y = x) \\
R \quad R_x & \quad R_y \quad R_y
\end{align*}
\]
(1) Let \( m \neq 0 \), then we have something
\[
\begin{align*}
\text{class from } x \to y \text{ must be } & \text{ } y \text{.}
\end{align*}
\]
(2) The domain of a nonconstant index 1
\[
\begin{align*}
x \quad y
\end{align*}
\]
\[ \text{Proof: } \#(S \cap P_{x,y}) = \#(K \cap D(\phi)) = \#(K \cap D(\phi)) \]
\[ (\exists S = K) \]

intersections look like:
\[ \begin{array}{c}
\text{w} \\
\text{z} \\
\text{D(\phi)} \\
\text{D(\phi)} \\
\end{array} \]

\[ S_q, \#(K \cap D(\phi)) = n_z(\phi) - n_w(\phi) \]

For \( x \in \alpha_i \cap \beta_j \), let \( \omega(x, K) = \text{winding # of } K \text{ around } x \).

\[ \text{e.g. } \begin{array}{c c c c}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
\end{array} \]

\[ \text{Lemma 2: } \#(S \cap P_{x,y}) = \sum_{x \in x} \omega(x, K) - \sum_{y \in y} \omega(y, K) \]

Hence, \( A(x) - A(y) = \sum_{x \in x} \omega(x, K) + C \)

Now \( \Delta_K(q) = \chi(\widehat{HF(K)}) \) and

\[ \Delta_K(q) (1 - q^{-1})^{-1} = \chi(\widehat{HF(T_x, T_y)}) = \sum_{x} (-1)^{x} q \]

\[ = \sum_{\sigma \in S_n} \epsilon(\sigma) \frac{n}{!} q x_{i, \sigma(i)} \cdot K \text{ constant } \]

\[ S_0, \Delta_K(q) = q^c (1 - q^{-1})^{(1-n)} \cdot \det(M_K(q)) \]

matrix of winding # monomials.
Other flavors of Heegaard Floer homology:

$HF^+, HF^-, HF^{\infty}$

Idea: now we count all $\phi$ with $m(\phi) = 1$ and keep track of $n_2(\phi)$

Remark: $m(\phi) + 2 \Rightarrow n_2(\phi) \geq 0$.

1. $HF^{\infty}$:
   
   \[ \mathcal{CF}^{\infty}(\mathcal{H}) = \mathbb{Z}[u,u^{-1}] \langle \pi_{\alpha} \cap \pi_{\beta} \rangle \text{ (as a } \mathbb{Z}[u]-\text{module)} \]

   \[ \gamma^\infty(x) = \sum \sum n_2(\phi)^2 \left( \frac{m(\phi)}{2} \right) U^{\frac{n_2(\phi)}{2}} \]

   \[ y \phi \in \pi_2(x,y) \]

   \[ m(\phi) = 1 \]

   \[ M(\Sigma) = 2 - 2g + g + g = 2. \]

Possible finiteness problem: If we count $\phi \in \pi_2(x,y)$, do we also count $\phi + k\Sigma$ for every $k \in \mathbb{Z}$?

**Lemma** $m(\phi + \Sigma) = m(\phi) + 2$ \hspace{1cm} (at most one of $\{\phi + k\Sigma\}$ has $m = 1$)

**Proof:** \[ m(\Sigma) = e(\Sigma) + n_4(\Sigma) + n_2(\Sigma) \]

\[ = (2 - 2g) + g + g = 2. \]

Grading: $HF^{\infty}(\mathcal{H},s)$ has $\mathbb{Z}/\delta(s)\mathbb{Z}$-grading relative to $\mathcal{H}$, given by $\text{gr}(x,y) = m(\phi) - 2n_2(\phi)$ \hspace{0.5cm} ($\phi \in \pi_2(x,y)$), with the convention that $U$ lowers grading by 2

\[ (\phi \in \pi_2(x,y), m(\phi) = 1, n_2(\phi) = k) \]

\[ \Rightarrow \text{gr}(x,y) = 1 - 2k \Rightarrow \text{gr}(x,U^ky) = 1 \]

\[ \text{So, } \text{gr}(U^kx,U^jy) = \text{gr}(x,y) + 2(k-j). \]
(2) $HF^+$

$CF^- \subset CF^\infty$ is the free $\mathbb{Z}[[u]]$-module generated by $\{ u^kx \mid x \in \mathcal{P}_a \cap \mathcal{P}_b, \ k \geq 1 \}$

$CF^+ = \frac{CF^\infty}{CF^-}$ is quotient complex

So, there is SES $0 \to CF^- \to CF^\infty \to CF^+ \to 0$

giving LES $\ldots \to HF^-(y) \to HF^\infty(y) \to HF^+(y) \to \ldots$

ex: $S^3$

There is also a SES $0 \to CF^- \xrightarrow{u} CF^- \xrightarrow{\text{"u=0"}} CF^+ \to 0$, gives LES $\ldots \to HF^+ \to HF^- \xrightarrow{u} HF^- \to HF^+ \to \ldots$

Remarks: ① $HF^+$ carries the least info

② $HF^\pm$ are needed to capture 4-mfld information from cobordisms.

③ $HF^\infty(y, b) = \mathbb{Z}[u, u^{-1}]$ when $b_1(y) = 0$

Thm (Lidman): $HF^\infty(y)$ is determined by cup products on $H^\bullet(y; \mathbb{Z})$. 
Similarly, we can define more general types of knot Floer homology:

**Definition:** Let $\mathcal{H} = (\Sigma, x, \beta, w, z)$ be a diagram for $K \subset S^3$.

\[
\text{CFK}^\infty(\mathcal{H}) = \mathbb{Z}[u, u^{-1}] \langle T_{\alpha} \cap T_{\beta} \rangle \ \text{(as a $\mathbb{Z}[U]$-module)}
\]

\[
\delta^\infty x = \sum_{\gamma \in T_{\alpha} \cap T_{\beta}} \sum_{\phi \in \pi_2(x, y)} \#(M(\phi)/\mathbb{Z}) U^{n_w(\phi)}
\]

\[
(\delta^\infty(U^k x) := U^k \delta^\infty x)
\]

**Maslov grading:**
\[
\phi \in \pi_2(x, y) \implies M(x) - M(y) = \mu(\phi) - 2n_w(\phi)
\]

\[
M(U^k x) = M(x) - 2^k
\]

**Alexander filtration:**
\[
A(x) - A(y) = n_z(\phi) - n_w(\phi)
\]

\[
A(U^k x) := A(x) - k
\]

**Notice:** If $U^k x$ is a term in $\delta^\infty x$, then
\[
M(x) - M(U^k x) = M(x) - M(y) + 2k = i - 2k + 2k = i.
\]

\[
M(x) - M(U^k y) = M(x) - M(y) + 2k = j - 2k + 2k = j.
\]

\[
A(x) - A(U^k y) = A(x) - A(y) + k = n_z(\phi) - n_w(\phi) + k = n_z(\phi) + k \geq 0.
\]

**Other flavors:**

1. $\text{CFK}^-(\mathcal{H}) = \mathbb{Z}[u^\pm] \langle T_{\alpha} \cap T_{\beta} \rangle$, $\delta = \delta^\infty$ (and same $M, A$), can view $\text{CFK}^\infty = \text{CFK}^- \otimes_{\mathbb{Z}[u^\pm]} \mathbb{Z}[u, u^{-1}]$ ($\mathbb{Z}[U]$-modules).

2. $g\text{CFK}^-(\mathcal{H}) = \text{associated graded complex of } \text{CFK}^-.$

Then $\hat{\text{CFK}} = g\text{CFK}^- / U g\text{CFK}^-$ ("set $U = 0").$
\( \alpha: \text{Trefoil knot} \)

Then \( \text{CFK}^\infty = \mathbb{Z}[u,u^{-1}] \langle a, b, c \rangle \)

\[ \partial^0 a = 0, \quad \partial^0 b = \mu c + a, \quad \partial^0 c = 0 \]

Grading: \( M(b) - M(a) = 1, \quad M(b) - M(c) = 1 - 2 = -1 \)

(forgetting \( z \) gives \( \hat{HF}(S^3) \cong \mathbb{C} \Rightarrow M(c) = 0 \) )

\[ A(b) - A(a) = 1 - 0 = 1, \quad A(b) - A(c) = 0 - 1 = -1 \]

\( \chi(\hat{HF}) = q^6 - 1 + q^{-1} \Rightarrow A(b) = 0 \)

A way to visualize \( \text{CFK}^\infty \) and \( \text{CFK}^- \):

1. In \( \mathbb{R}^2 \), we plot a generator \( U^j x \) with \( A(U^j x) = k \) at the point \((j, k)\):

\( \text{CFK}^\infty \)

\( \text{gCFK}^- \)

\( \hat{HF} \)

\( \text{CFK}^- \)

Note: In general,

\( H_*(\text{CFK}^\infty) = \mathbb{Z}[u,u^{-1}] \)
Recall: Let \( \mathcal{H} \) be strongly admissible.

- \( \text{CF}^\infty(\mathcal{H}) \) is generated over \( \mathbb{Z}[u, u^{-1}] \) by \( T_\alpha \cap T_\beta \).
- \( \text{CF}^- (\mathcal{H}) \subset \text{CF}^\infty(\mathcal{H}) \) is generated over \( \mathbb{Z} \) by
  \[ \{ u^k x \mid x \in T_\alpha \cap T_\beta, k \geq 1 \} \] (a subcomplex)
  (think of \( \text{CF}^- \), \( \text{CF}^\infty \) as \( \mathbb{Z}[u] \)-modules).
- \( \text{CF}^+(\mathcal{H}) = \text{CF}^\infty(\mathcal{H}) / \text{CF}^-(\mathcal{H}) \)

There are two notable sequences:

1. \[ 0 \rightarrow \text{CF}^- (\mathcal{H}) \xrightarrow{\ell} \text{CF}^\infty(\mathcal{H}) \xrightarrow{\pi} \text{CF}^+(\mathcal{H}) \rightarrow 0 \]
   induces
   \[ \text{HF}^- (\mathcal{H}) \xrightarrow{i^*} \text{HF}^\infty(\mathcal{H}) \]
   \[ \begin{array}{c}
   \downarrow \\
   \pi_* \\
   \end{array} \]
   \[ \text{HF}^+(\mathcal{H}) \]
   \[ \text{HF}_{\text{red}} (\mathcal{H}) := \text{coker}(i^*) \cong \ker(i_*); \quad \text{HF}_{\text{red}} = \text{HF}^+ \cap \text{HF}^- \]

2. \[ 0 \rightarrow \text{CF}^- (\mathcal{H}) \xrightarrow{u^*} \text{CF}^0(\mathcal{H}) \xrightarrow{u=0} \hat{\text{CF}}(\mathcal{H}) \rightarrow 0 \]
   induces
   \[ \text{HF}^- (\mathcal{H}) \xrightarrow{\wedge} \text{HF}^0(\mathcal{H}) \]
   \[ \begin{array}{c}
   \downarrow \\
   \hat{\text{HF}}(\mathcal{H}) \\
   \end{array} \]
   (This allows one to compute \( \hat{\text{HF}} \) from \( \text{HF}^- \))
ex: \( S^3(\mathbb{R}^n\mathrm{Teich}) = \mathbb{Z}_n \)  

\( \mathbb{Z}_n \) is the \( \mathbb{Z} \) \( H_x \times S^3 \) \( \sum (2,3,6n-1) \) (A "Brieskorn sphere")

\[ \text{HF}^+_k(\mathbb{Z}_n) = \begin{cases} 
\mathbb{Z} & k \text{ even, } k \geq 0 \\
\mathbb{Z}^n & k = -2 \\
0 & \text{else}
\end{cases} \] (won't prove)

HF\(^-\) \hspace{1cm} HF\(^\infty\) \hspace{1cm} HF\(^+\)

Now \( \text{HF}_{\text{red}}(\mathbb{Z}_n) \cong \mathbb{Z}^{n-1} \)

**Rank:**
1. One can view \( \text{HF}_{\text{red}}(Y) \) as \( \text{HF}^+(Y)/\text{Im}U \), where \( k \) is sufficiently large.
2. One can view \( \text{rk} \text{HF}_{\text{red}}(Y) \) as a measure of complexity of \( Y \). When \( Y \) is a \( \mathbb{Z}H_x^{S^3} \) (e.g. \( \mathbb{Z}_n \) above),

\[ \text{HF}^+(Y) = \mathbb{Z}[u,u^{-1}] \oplus \text{HF}_{\text{red}}(Y) \]
One can define maps on $\overset{\wedge}{HF}$, $HF^-$, etc. associated to cobordisms between closed 3-manifolds. Any cobordism \[
\begin{array}{c}
y_1^3 \\
\downarrow
\end{array}
\begin{array}{c}
y_2^3
\end{array}
\]

of \( W^m \) can be decomposed into 1, 2, and 3-handle attachments.

Recall: Attaching a \( m \)-dimensional \( k \)-handle to \( W^m \) consists of attaching a \( D^k \times D^{m-k} \) to \( \partial W \)

\[ \partial D^k \times D^{m-k} = S^{k-1} \times D^{m-k} \subset \partial (D^k \times D^{m-k}) \text{ via an embedding } \varphi : S^{k-1} \times D^{m-k} \to \partial W. \]

E.g. \( (m=4) \)

\[ \begin{align*}
k=1: & \quad D^1 \times D^3 \text{ attached along } S^0 \times D^3 : \quad \overset{\bullet}{\circ} \\
k=2: & \quad D^2 \times D^2 \text{ along } S^1 \times D^2 : \\
& \quad \text{nbd of a knot, clenched NFA \rightarrow} \quad \overset{\circ}{\circ}
\end{align*} \]

Will use \[
\begin{array}{c}
y_1^3 \\
\downarrow
\end{array}
\begin{array}{c}
y_2^3
\end{array}
\]

\( W_i^0 \) handle attachment to \( Y \times [0,1] \)

along \( Y \times \{\varepsilon\} \)

to define chain map \( CF(Y_1) \to CF(Y_2) \)

(i) \( i=1 \):

\[
Y_1 \times \varepsilon \rightarrow \overset{\circ}{\circ} \]

(\( Y_1 \times \varepsilon \#: (S^2 \times S^1) \))

(on \( Y_1 \times \varepsilon \), we remove \( S^0 \times D^3 \))

and attach the other piece

of \( \partial (D^1 \times D^3), D^1 \times S^2 \)

\[ F_{W_1}^0 : HF^0(Y_1) \to HF^0(Y_1 \# (S^2 \times S^1)) = HF^0(Y_1) \otimes HF^0(S^2 \times S^1), \]

\[
x \mapsto x \otimes \Theta
\]

(\( \partial (x \otimes \Theta) = \partial x \otimes \Theta + x \otimes \delta \Theta \)

\( \delta (x \otimes \Theta) = x \otimes \Theta + x \otimes \delta \Theta \))
\[ i = 3: \quad \text{Essentially the opposite of attaching a 1-handle: } \ Y = \ Y_1 \# (S^2 \times S^1), \] and define
\[ F^*: \ HF^0(Y \# (S^2 \times S^1)) \to HF^0(Y) \text{ via} \]
\[ \begin{align*}
x \circ \emptyset & \mapsto 0 \\
x \times \eta & \mapsto x
\end{align*} \]

\[ i = 2: \quad \text{Two-handles are more complicated:} \]
\[ \begin{align*}
& (a) \text{ there are many knots in } Y \setminus \Sigma, \\
& (b) \text{ there are many ways to embed } S^1 \times D^2 \text{ as } N(K)
\end{align*} \]

**Knot surgery:**

Choose a knot \( K \subset Y \) has a well-defined meridian \( \mu \subset N(K) \)

Let \( M \) primitive in \( H_1(\partial N(K)) \), trivial in \( H_1(N(K)) \)

A **longitude** \( \lambda \) for \( K \) is a s.c.c. in \( \partial N(K) \)

\[ \text{s.t. } \ (\mu \cap \lambda) = -1 \text{. We call } (K, \lambda) \text{ a framed knot.} \]

(fixing \( \lambda \), the set of longitudes is \( \{ \lambda + \eta \mu \}_{\eta \in \mathbb{Z}} \))

**Def:** Let \( (K, \lambda) \) be a framed knot in \( Y \). Then

\[ Y_{\lambda}(K) := \left( Y \setminus N(K) \right) \cup_{h} (S^1 \times D^2), \] where

\( h: \partial N(K) \to S^1 \times S^1 = \partial(S^1 \times D^2) \) s.t. \( h(\lambda) \) bounds a disk:

\[ Y_{\lambda}(K) \text{ is } \lambda \text{-framed surgery on } K \subset Y. \]
Fact: The cobordism $W_2$ is determined by $(K, \gamma)$ (denote it by $\gamma_{W_2(K)}\gamma(K)$).

More generally, we can analogously define $\gamma_Y(K)$ for any homotopically non-trivial curve $Y \subset N(K)$ (not just longitudes) - but then there isn’t a canonical $W_2(K)$.

Def: Let $K$ be nullhomologous. Then there is a unique longitude $\lambda_0$ which is null-homologous in $\gamma_Y \setminus N(K)$. Call it the Seifert-framing.

Now if $Y = pl + g \lambda_0$, we say $\gamma_Y(K)$ is "$\frac{p}{q}$-surgery on $K \subset Y$", denote it by $\gamma_{\frac{p}{q}}(K)$.

Exercise: $H_1(\gamma_{\frac{p}{q}}(K)) \cong \mathbb{Z}/p\mathbb{Z}$ if $\frac{p}{q} \in \mathbb{Q}$

ex: 1) $\gamma_0(K) = Y$ for any $K$

2) $S_{\gamma_0}^3(U) = S^3$

3) $S_{\frac{p}{q}}^3(U) = L(p, q)$ (lens space)

4) $S_{\frac{0}{q}}^3(U) = S^2 \times S^1$

5) $S_1^3(RT_trefoil) = \Sigma(2, 3, 5)$, Poincaré homology sphere.
Fix \((K, \lambda) \subset \mathbb{Y}^3\), and let \(Y_0 := Y^1_A(K)\) \(Y_1 := Y^1_{A+\mu}(K)\)

e.g. \(Y = S^3\), \(Y_0 = S^3_p(K)\), \(Y_1 = S^3_{p+1}(K)\) \((\lambda = p\mu + \lambda_0)\)

Note: We call \((Y, Y_0, Y_1)\) a triad of \(3\)-mfds

Exercise: A cyclic permutation of a triad is another triad.

Thm \(\circled{3}\) (Oz-Sz.) If \((Y, Y_0, Y_1)\) is a triad, then there are HES.

\[ \cdots \to HF^0(Y) \to HF^0(Y_0) \to HF^0(Y_1) \to \cdots \]

(as modules)

How do the maps in the sequence work?

1. Choose a Heegaard diagram \(H = (\Sigma, \alpha, \beta, z)\) for \(Y\) such that
   - \((\Sigma, \alpha, \beta \backslash \beta_3, z)\) is a diagram for \(Y \setminus N(K)\)
   - \(\beta_3\) is a meridian for \(K\).

2. Let \(Y_i\) be isotopic to \(\beta_i\) for \(i < g\) and \(\delta_g\) represent \(\bar{\lambda}\).
   Then \((\Sigma, \alpha, \delta, z)\) is a diagram for \(Y_0 = Y_1(K)\).

3. \(\delta_g\) is isotopic to \(\beta_i\) for \(i < g\), \(\delta_g\) represents \(\lambda + \mu\)
   Then \((\Sigma, \alpha, \delta, z)\) is a diagram for \(Y_1 = Y_{\lambda+\mu}(K)\).

Note: \(Y_{\delta_g} \cong Y_{\delta} \cong Y_{\delta_\delta} \cong \#(S^2 \times S^1)\)
Recall we had
\[
\mathcal{F}_\alpha \rho \gamma \rightarrow \hat{\mathcal{F}}(Y_{\alpha \rho \gamma}) \rightarrow \hat{\mathcal{F}}(Y_{\alpha \rho}) \rightarrow \hat{\mathcal{F}}(Y_{\alpha})
\]

Then we have
\[
\hat{\mathcal{F}}(y) \overset{\mathcal{F}}{\longrightarrow} \hat{\mathcal{F}}(Y_0) \quad (\star)
\]

where
\[
\hat{\mathcal{F}}(y) := \hat{\mathcal{F}}_{\alpha \rho \gamma} (y \otimes \Theta_{\alpha \rho \gamma})
\]
\[
\hat{\mathcal{F}}_0(y) := \hat{\mathcal{F}}_{\alpha \rho \delta \gamma} (y \otimes \Theta_{\alpha \rho \delta \gamma})
\]
\[
\hat{\mathcal{F}}_1(y) := \hat{\mathcal{F}}_{\alpha \delta \beta \gamma} (y \otimes \Theta_{\alpha \delta \beta \gamma})
\]

Ingredients for proving \((\star)\) is exact:

1. \(F_0 \circ \hat{\mathcal{F}}(y) = \hat{\mathcal{F}}_{\alpha \delta \beta \gamma} \left( y \otimes \hat{\mathcal{F}}_{\delta \beta \gamma} \left( \Theta_{\alpha \delta \beta \gamma} \otimes \Theta_{\delta \beta \gamma} \right) \right) \]

\[
= \hat{\mathcal{F}}_{\alpha \delta \beta \gamma} \left( y \otimes \hat{\mathcal{F}}_{\delta \beta \gamma} \left( \Theta_{\alpha \delta \beta \gamma} \otimes \Theta_{\delta \beta \gamma} \right) \right) = 0 \quad \text{("associativity", exercise)}
\]

2. It turns out that this chain complex is acyclic.

\((\Rightarrow \: \star \: \text{is exact})\)
Cobordisms:

By composing handle attachment maps, we can obtain from \( W^4 \) an induced map
\[
\begin{array}{ccc}
W^3 & W^3 \\
\downarrow & \downarrow \\
Y_1^3 & Y_2^3
\end{array}
\xrightarrow{F^0_{W_S}} \text{HF}^0(Y) \rightarrow \text{HF}^0(Y_2)
\]

These give \( \text{HF}^0 \) the structure of a \("(3+1)\)-dimensional TQFT," as
\[
\begin{array}{ccc}
W^3 & W^3 \\
\downarrow & \downarrow \\
Y_0^3 & Y_1^3 & Y_2^3
\end{array}
\xrightarrow{F^0_{W_2 W_1}} F^0_{W_2} F^0_{W_1}
\]

Given \( s \in \text{Spin}^c(W) \), one can "restrict" to \( Y_i = s_i \in \text{Spin}^c(Y_i) \).
Then we have maps
\[
F^0_{W_S} : \text{HF}^0(Y, s_i) \rightarrow \text{HF}^0(Y_2, s_2)
\]

From now on, assume \( Y_i \) are \( QH_* S^3 \)'s (i.e. \( H_2(Y_i) = 0 \))
Then \( \text{HF}(Y_i, s_i) \) has relative \( \mathbb{Z} \)-grading, and \( F_{W_S} \) is homogeneous w.r.t it. We can beef this up a bit.

Intersection Forms:

Let \( M^4 \) be 4\text{d}, oriented, and assume \( \partial M \) is \( QH_* S^3 \).
Then there is an intersection form
\[
Q_M : H^2(M; \mathbb{Q}) \otimes H^2(M; \mathbb{Q}) \rightarrow \mathbb{Q}
\]
given by
\[
Q(\xi \otimes \eta) := \langle \xi \cup \eta, [M] \rangle
\]
\[
( x^2 := Q(x,x) \text{ for shorthand})
\]
Thm: Let $t \in \text{Spin}^c(Y)$ be a torsion element, i.e. assume $c_1(t) = 0$ (e.g. every $s$ is torsion if $Y$ is $\mathbb{QHS}^3$).

Then there is a unique $\mathbb{Q}$-valued lift $\tilde{g}_r$, $\tilde{g}_r$, of the relative $\mathbb{Z}$-grading on $HF^0(Y, t)$ s.t:

(i) $\tilde{g}_r(S^3) \in \mathbb{Z}$ is supported in $\tilde{g}_r = 0$.

(ii) $L$, $\hat{L}$, $\Pi$ are degree-preserving in:

$$HF^-(Y, t) \xrightarrow{\hat{L}} HF^0(Y, t) \xrightarrow{\Pi} HF^+(Y, t)$$

and

$$HF^-(Y, t) \xrightarrow{\hat{g}_r} HF^0(Y, t) \xrightarrow{\Pi} HF^+(Y, t)$$

(iii) If $W^4$ is a cobordism from $Y_1$ to $Y_2$ and $\beta \in HF^0(Y_1, t_1)$,

$$\tilde{g}_r(F_{W, s}(\beta)) - \tilde{g}_r(\beta) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}$$

for any $s$ with $S|_{Y_1} = t_1$.

Def: Let $Y$ be a $\mathbb{QHS}^3$ and choose $t \in \text{Spin}^c(Y)$.

The correction term for $(Y, t)$ is

$$d(Y, t) := \min \left\{ r \in \mathbb{Q} \mid \exists x \text{ homogeneous non-torsion} \ x \in \text{Im}(\Pi : HF^0(Y, t) \to HF^+(Y, t)) \right\}$$

Facts: $d(-Y, t) = -d(Y, t)$

$d(Y \# Z, t \# u) = d(Y, t) + d(Z, u)$
Thm \(O^2-S^2\): Let \(Y^3\) be a QHS, and fix \(z \in \text{Spin}^c(Y)\). Then for each smooth, negative-definite 4-manifold \(W\) with \(\partial W = Y\), and for each \(s \in \text{Spin}^c(W)\) with \(s|_Y = z\),
\[
c_1(s)^2 + b_2(W) \leq 4d(Y, z)
\]

**Idea:** Explicitly study the cobordism \(S^3 \to Y^3\)
and find a homogeneous element \(x \in HF^+(S^3)\)
\(s.t.\ \tilde{\gamma}(F_{w,s}(x)) = d(Y, z).\) Therefore,
\[
d(Y, z) - \tilde{\gamma}(F_{w,s}(x)) - \tilde{\gamma}(x) = \frac{c_1(s)^2 - 2 \chi(W) - 3 \sigma(W)}{4}
\]
\((\tilde{\gamma}(x) = 0)\)
But \(\sigma(W) = -b_2(W)\) and \(\chi(W) = b_2(W)\)
\[
\Rightarrow d(Y, z) \geq \frac{c_1(s)^2 + b_2(W)}{4}
\]

If \(Y\) is a ZHS, then the intersection form of \(X\)
takes values in \(\mathbb{Z}\) and is unimodular.

Thm (Elkies): Let \(Q: V \otimes V \to \mathbb{Z}\) be a neg. def unimodular bilinear form over \(\mathbb{Z}\). Then
\[
0 \leq \max_{x \in X(Q)} \left\{ Q(x, x) \right\} + n
\]
with equality \(\iff Q\) is diagonalizable over \(\mathbb{Z}\).
\* \(X(Q)\) is the set of characteristic vectors for \(Q\), i.e., the set
\[
\left\{ x \in V \mid Q(x, y) \equiv Q(y, y) \pmod{2}, \forall y \in V \right\}
\]

**Corollary:** Let \(Y^3\) be a \(\mathbb{Z}HS^3\) with \(dX^4 < Y\), where \(X\) is negative definite. Then
\[
Q_x(x, x) + b_2(X) \leq 4d(Y)
\]
for each \(x \in X(Q_x)\).

**Proof:** Each characteristic vector is \(\psi(s)\) of some \(s \in \text{Spin}^c(X)\).

**Corollary:** Let \(Y^3\) be \(\mathbb{Z}HS^3\) with \(d(Y) < 0\). Then there is no neg. def. \(X^4\) with \(\exists X = Y\).

**Example:** \(-\Sigma(p, q, pqn - 1) = S^3_{\frac{1}{n}}(T_{p, q})\) torus knot

Now \(T_{p, q}\) can be unknotted by changing positive crossings to negative, and so \(-\Sigma(p, q, pqn - 1)\) bounds pos. def. \(X^4\). However, \(d(S^3_{\frac{1}{n}}(T_{p, q})) < 0\), and so it doesn't bound neg. def. \(X\).
**Def.** Let \((Y_i, t_i) (i=1,2)\) be a pair of QHSS\(^3\)'s \(Y_i\) with \(t_i \in \text{Spin}^c(Y_i)\).

We say \((Y_i, t_i)\) are \(Q\)-homology cobordant if there is a cobordism \(Y_1 \rightarrow W \rightarrow Y_2\) s.t.\[
H_\ast(W, \mathbb{Q}) \cong H_\ast(S^3 \times I, \mathbb{Q})\]
and \(s \in \text{Spin}^c(W)\) with \(s|_{Y_i} = t_i\).

**Prop.** When \((Y_i, t_i)\) are QHCBs, \(d(Y_i, t_1) = d(Y_2, t_2)\).

**Cor.**

1. \(d(S^3(K))\) is a smooth concordance inv \(2\).

   (studied by T. Peters)

2. \(s(K) = 2d(\Sigma(K), \text{Spin}^c)\) is also double branched cover of a distinguished Spin\(^c\)-structure

(in fact, both are homs. \(C_{\text{smooth}} \rightarrow \mathbb{Z}\)).